כיוון בית הכנסת והתפילה.

היה מקובל בזמן המשנה והגמרא שכיוון בית הכנסת והתפילה, הוא כנגד ירושלים והמקדש והקודש הקודשים. הדבר כל כך ברור שעוד היום, כאשר ארכאולוגים מגלים בניין תפילה ישן, הם מכריעים לפי כיוון של הבניין, אם הוא היה בית הכנסת או כנסיה.
אם הרחבת תפוצות ישראל, היהודים נעמתו לבעיית כיוון בתי כנסיות לעבר ירושלים כאשר הם הבינו שהארץ כדורית. ולכן אי אפשר לקבל עוד את הקירוב המישורי.
אנחנו מראים ששני פתרונות היו בהתחרות כדי למצוא פיתרון לבעיית כיוון בית הכנסת והתפילה. הפיתרון הראשון, הקו הגיאודטי או קו של המעגל הגדול העובר על המקום בו אנו מתחשבים, וירושלים. רבנים חשובים תמכו בפיתרון זה: רבי אביעד שר שלום בזיליא, רבי יעקב עמדן, רבי ישראל זמושק והרב שניאור זלמן מליאדי.
הפיתרון השני הוא הקו של הלוקסודרומיה העוברת על המקום בו אנו מתחשבים וירושלים. יחסו את הפיתרון הזה, זמן ארוך אחר מותו, לרבי מרדכי יפה, בעל הלבושים. אבל הדבר מוטל בספק.
אנחנו מפרשים את המקורות של שני הפתרונות . אנחנו מסבירים ביסודיות את שני הפתרונות בפרספקטיבה הסטורית. אנחנו מראים, שלאמיתו של דבר, רק הפיתרון הראשון של הקו הגיאודטי נכון.
הפיתרון השני מספק את "השכל הישר" של פשוטי עם וגם של תלמידי חכמים, בלי שום השכלה מתמטית. ובאמת, למעשה העולם בטוח שצריכים להתפלל בכיוון מזרח, בדיוק בניגוד לפסקו של רב ששת.

The Orientation of the Synagogue. The Prayer Direction.

abstract

It was generally accepted in the time of the Mishnah and the Gemara that the synagogues and the prayer must be directed towards Jerusalem and more precisely towards the Temple and the Holy of the Holy.
This is so true that still today when archeologists discover an ancient worship building they make the distinction between ancient synagogues and churches according to their orientation. Churches were directed eastward while synagogues were directed toward Jerusalem.
With the extension of the Diaspora, Jews faced the problem of the orientation of their synagogues toward Jerusalem when they realized that the earth is spherical and that the planar approximation could not more be accepted.
We show how two different solutions were in competition in order to solving the problem. The first was the geodesic line or the orthonome joining the considered locus to Jerusalem i.e. the great circle intersection of the terrestrial sphere with the plane defined by the two former points and the center of the earth. This solution was championed by R. Solomon Aviad Sar Shalom Basilea, R. Jacob Emden, R. Israel Zamosc and R. Shneor Zalman of Liady. The second was the loxodrome i.e. the course of constant bearing, joining the considered locus to Jerusalem. This solution was ascribed posthumously to R. Jaffe in his *Levush*, but this attribution remains uncertain and questionable.
We explain the nature and the origin of these two solutions in their historical perspective and we explain the theory of each of these solutions.
We show that in fact, only the first solution of the geodesic line is valid. The second solution satisfies the common sense and the naïve appreciation of people without mathematical education. In fact people are convinced that we must pray eastward, just in contradiction with the dictum of Rav Sheshet.

The Orientation of the Synagogue. The Prayer Direction.

1. Introduction.

There is no limit to the progress. Today it is possible to find the orientation of the Jewish prayer on the web site *Kosher Java*. But surprise! It doesn't give you the solution but it proposes you two solutions. If you are in North America these two solutions are notably different. The one gives a direction eastward with a slight deviation to the south while the second gives a prayer direction eastward with a serious deviation to the north. This is the malediction of the Jewish people in its exile in accordance with the dictum of Rav and the Sages of Yavneh: "The Torah will be forgotten by Israel….they will wander to find the words of God and they won't find it, meaning that they won't find a precise and clear-cut ruling and teaching in one place שלא ימצאו הלכה ברורה[1] ומשנה ברורה במקום אחד.[2] And in fact in most aspects of Jewish life we find always contradictory opinions. Even in practical life there are often two divergent uses.

It is however surprising that in an issue, which seems more mathematical than rabbinical,[3] we still have two divergent solutions. It is still more surprising that there is no preference. The solution is left to the choice of the user.[4]

The aim of this paper is to analyze this interesting problem. Its historical survey will learn us a lot about the slow development of exact sciences and mathematics in the Jewish rabbinical elite until the modern time.[5]

We will show that the issue of the prayer direction has only one solution. It is the tangent in the considered location to the great circle[6] passing through that location and Jerusalem. The second proposed solution is that of the rhumb[7] line passing through the considered location and Jerusalem. It is the result of an historical misunderstanding of the maps and atlases which appeared when the Jews immigrated en masse to the new world during the nineteenth century. We find the same discussion and the same mistake in the Muslim world; but they at least succeeded to solve definitively the problem on a scientific basis.

Today it seems that the solution based on the rhumb line is gaining in popularity. Some *haredi* rabbis seem to champion this last solution. They rewrite and reinterpret the

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[1] בטעמים שלא יהא בה מחלוקת, רש"י. This is certainly the origin of the title of the commentary of R. Israel Meir ha-Kohen Kagan (1838-1933) on *Shulhan Arukh Orah Hayim*. This commentary should fill this gap.

[2] B. Sabbath 128b-129a.

[3] The divergence of the measure of tefah, the amah and the mile is of a different nature, it is a problem of tradition. See my paper: Talmudic Metrology I, The Mile as Unit of Length, BDD 19, January 2008

[4] At least according to the website Kosher Java. The study of the *Shulhan Arukh* does not allow making a choice between both solutions.

[5] In fact there are some rulings which require a good mathematical culture which goes far beyond the knowledge of the average rabbi.

[6] See appendix 4: Orthonome and loxodrome.

[7] See appendix 4: Orthonome and loxodrome.

commentary of R. Mordekhai Jaffe. It is likely that the latter had never in mind this new approach.
Not only both solutions are presented on the same level on a web application but some *haredi* rabbis rule clearly in favor of the rhumb line.
We note indeed that in the last years there appeared advertisements on the web for a *kosher compass* aimed at the individual determination of the prayer direction. According to its website, four current rabbinical authorities, namely R. Ya'akov Perlow, R. Moshe Halbershtam, R. Moshe Sternbuch and R. Yosef Lieberman gave their approbation to this kosher compass. Now this device works i.e. the producers voluntary calibrated the device, according to the principle of the rhumb lines. Therefore the approbation of the device represents also an indirect ruling in favor of the principle of the rhumb lines.

2. The exact prayer direction according to the great circle.

If the earth was flat the problem of praying toward Jerusalem would not raise any difficulty. We would pray in the direction of the straight line joining the considered location and Jerusalem.
Now on a spherical earth the straight lines become great circles. Indeed the natural way to bend a straight line in order to compel it through two locations, for example New York and Jerusalem, is to bend it in one direction, while beholding its planar shape. If we impose to behold a symmetrical position to this plane with regard to the sphere of the earth, the plane must contain the center of the sphere. The straight line becomes then a great circle of the sphere. This great circle passing through New York and Jerusalem is the shortest distance between these two towns. This great circle presents a simple curvature, in the plane of the circle. It seems then genuine to consider that the prayer, in New York, propagates along the great circle between New York and Jerusalem.
In the commentary *Perisha* on *Tor Orah Hayim* 94, the author compares the propagation of the prayer to the trajectory of an arrow which moves straight to the target.
If we imagine a canon placed on top of a high mountain in a point A, firing an arrow or a ball with an initial velocity perpendicular to the radius OA and therefore tangential to the surface of the sphere passing through A and concentric to the earth, with center O. If it is given a low initial speed, the ball travels in an approximately parabolic path (ignoring the air resistance) in the plane defined by the radius OA and the vector celerity C. If the initial speed is high enough, the ball travels right around the earth, back to the starting point (thus the path of the satellite never intersects the surface of the earth and it never lands). Of course we ignore and neglect any retarding force due, for example, to the atmosphere of the earth. The ball of mass m covers thus a stable circular orbit around the earth of mass M. According to the second law of Newton F = m*a, the inward force of gravity provides the centripetal acceleration.

$$\frac{G\,m\,M}{r^2} = \frac{m\,v^2}{r} \quad \text{hence: } V_{\text{orbit}} = \sqrt[2]{\frac{G\,M}{r}}$$

We see thus that any projectile, ball or arrow, sent with an initial velocity tangent to the sphere concentric to the earth, covers a path situated in the plane of a great circle of the earth. When this initial velocity reaches the size of v_{orbit}, then the trajectory becomes a great circle which is a circular and stable orbit and the projectile becomes a satellite of the earth. We see thus that the great circle has a physical signification in the propagation of a projectile and it makes sense to assimilate the trajectory of our prayers to a great circle or geodesic line, see figure 1.

A more geometrical argumentation could be as follows. Let us consider a transparent earth with a lightened center. The night observer in New York or an external observer on a satellite

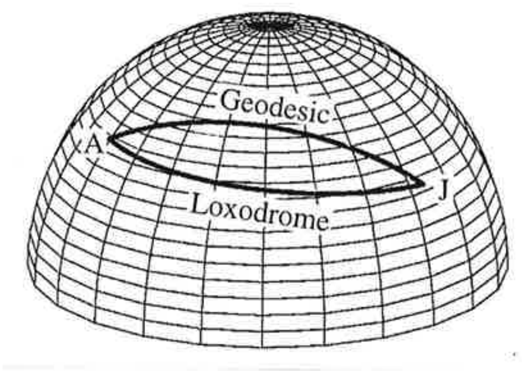

Figure 1: The great circle, also called geodesic line and orthonome, and the rhumb line or loxodrome joining the location A (for example New York) to the point B (for example Jerusalem). On our drawing the prayer direction, in the location A, is northeastward according to the great circle theory and southeastward according to the loxodrome theory.

will then see or imagine the straight line of light crossing the earth and joining New York to Jerusalem specially lightened for the circumstance.
Then the observer in New York or on the satellite will naturally consider that the straight path on the surface of the earth between New York and Jerusalem is the projection of the lightened straight line New York – Jerusalem on the surface of the earth from its center.
For the same observers, the rhumb[8] line joining New York to Jerusalem would appear as a segment of a spherical helix and they would consider that this line is a very strange and odd solution for the straight path joining New York to Jerusalem.
Another consideration allows finding the prayer direction while respecting the concept of straight propagation of the prayer. If we consider the linear propagation of the prayer along the tangent to the great circle in the considered location, this tangent in New York is in the plane of the great circle passing through New York and Jerusalem and it intersects the zenithal direction of Jerusalem.
According to the symbolic adopted in the Talmud,[9] the prayer pronounced in the Temple passes through the Gates of the sanctuary,שערי היכל, and then they pass through the gates of the sky,שערי שמים. According to our scheme, the prayers coming from New York reach directly the sky on the zenithal direction of Jerusalem after escaping and bypassing the gates of the sanctuary and the gates of the sky which apparently work according to the local time of Jerusalem. You might consider this scheme as naïve or stupid, but it has some other merit. Indeed if the prayers coming from New York had to pass through the Temple they would arrive at a time of closure of the gates of the sanctuary and of the sky and they should be stored until the next day. Such a local was not described in *Massekhet Midot*. According to the proposed scheme, not only we justify the principle of the determination of the direction toward Jerusalem but we explain that the sky is always open to accept in real time prayers coming from everywhere. Thus by a purely straight linear propagation along the tangent in New York to the great circle passing through New York and Jerusalem, prayer can reach the sky on the zenithal direction of Jerusalem.

[8] See appendix 4: Orthonome and loxodrome.
[9] For the prayer of Ne'ila and the closure of the gates: see B. Ta'anit 26b, B. Yoma 87b and Y. Ta'anit IV, 67c. See also Rambam Hikhot Tefila 1: 7 and 3: 6 with Hagahot Maimoniot, note [5].

For all these reasons the direction from New York to Jerusalem is given by the great circle joining these two locations. Especially the mechanical analogy comparing the prayer to an arrow, shows convincingly that it makes sense to consider that facing Jerusalem in New York, means directing oneself along the tangent to the great circle passing through these two towns.

Champion the theory of the rhumb line for the direction between New York and Jerusalem means that we want to ship our prayers to Jerusalem along a loxodrome, a curve presenting a double curvature. It is the course followed by the ships during the sixteenth, the seventeenth and the eighteenth centuries, when the navigators were not yet able to estimate correctly their longitude and were afraid to get lost. They followed rhumb lines allowing navigation without changing the direction as measured relative to true north. This path is crossing the meridians at the same angle. This path was fitted for the navigation of ships but it is not adapted to be the propagation path of our prayers.

3. Calculation of the prayer direction according to the great circle.

1. The classical solution, see figure 2.[10]

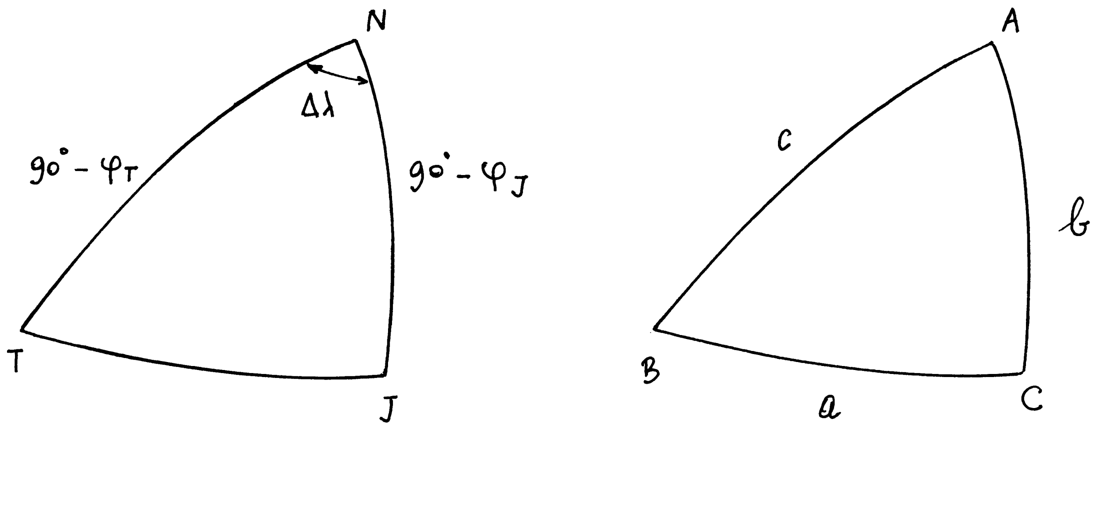

Figure 2: Spherical triangle defined by the North Pole, Jerusalem J and the considered location T. The left triangle with the summits A, B and C allows using the classical formulas. Thus J=C, T=B, N=A and c =90° - φ_T, the complement of the latitude of the locus, b = 90° - φ_J, the complement of Jerusalem's latitude.

The formulas of Napier[11]:

[10] This solution is the classical solution taught from the beginning of the 17th century, when these formulas were published by John Napier in 1614 until today. These formulas were preferred to the fundamental formulas because they are logarithmic and they allow a precise manual calculation with a logarithm table. The formulas of Delambre could also be used. However the analogies of Delambre were published only at the beginning of the 19th century and therefore they are not considered as the classical solution.

[11] See appendix 2: Formulas of the spherical trigonometry. These formulas allow calculating the angles B and C when we now the two opposite sides and the inner angle.

$$tang\ \frac{B+C}{2} = \frac{\cos\frac{b-c}{2}}{\cos\frac{b+c}{2}}\ cotg\ \frac{A}{2}$$

$$tang\ \frac{B-C}{2} = \frac{\sin\frac{b-c}{2}}{\sin\frac{b+c}{2}}\ cotg\ \frac{A}{2}$$

Example. T = New York $\lambda_T = 73.8°$ W $\varphi_T = 40.8°$ N
J = Jerusalem $\lambda_J = 35.2°$ E $\varphi_J = 31.8°$ N
c = 90° – 40.8° = 49.2°, b = 90°– 31.8° = 58.2°, b – c = 9° and b + c = 107.4°

$$tang\ \frac{B+C}{2} = \frac{\cos 4.5°}{\cos 53.7°} cotg\ 54.5°$$

$$tang\ \frac{B-C}{2} = \frac{\sin 4.5°}{\sin 53.7°} cotg\ 54.5°$$

Hence B + C = 100.44°
B – C = 7.94°
B = 54.19°
C = 46.25°

2. Other methods.

1. We use the fundamental formulas:[12]

$$\cos a = \cos b * \cos c + \sin b * \sin c * \cos A$$

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

Example. T = New York $\lambda_T = 73.8°$ W $\varphi_T = 40.8°$ N
J = Jerusalem $\lambda_J = 35.2°$ E $\varphi_J = 31.8°$ N

$$\cos a = \cos 58.2° * \cos 49.2° + \sin 58.2° * \sin 49.2° * \cos 109° = 0.13$$

Hence a = 82.25° and the length in km of the arc a, which we represent by la, is then la = 2 π * 6371.221 * 82.25/360 = 9146 km.

$$\frac{\sin 109°}{\sin 82.25°} = \frac{\sin B}{\sin 58.2°} = \frac{\sin C}{\sin 49.2°}$$

Hence B = 54.19°
C = 46.25°

[12] See appendix 2: Formulas of the spherical trigonometry.

2. If we don't want to know a, the distance between the two towns, we can use one of the six formulas of the cotangents.[13]

The following formula allows calculating B when we know b, c and A.

$$cotg\ b * \sin c = \cos c * \cos A + \sin A * cotg\ B$$

Which we can write on the following way:

$$\tan \varphi_J * \cos \varphi_T = \sin \varphi_T * \cos \Delta\lambda + \sin \Delta\lambda * \text{cotg } B$$

Example 1: T = New York $\lambda_T = 73.8°$ W $\varphi_T = 40.8°$ N
J = Jerusalem $\lambda_J = 35.2°$ E $\varphi_J = 31.8°$ N

$$tang\ 31.8° * \cos 40.8° = \sin 40.8° * \cos 109° + \sin 109° * cotg\ B$$

Hence: 0.47 – (– 0.21) = 0.95 * cotg B; tang B = 1.39 and B = 54.19°.

Example 2: T = Bagdad $\lambda_T = 44.4°$ E $\varphi_T = 33.4°$ N
Tang 31.8° cos 33.4° = sin 33.4° cos 9.20° + sin 9.20° cotg B. Hence B = 99.16°
The prayer direction is thus westward with a slight deviation of 9.20° southward. This slight deviation southward was sufficient to be noticed in the Talmud by the word: אדרימו.[14]

3. Solution of the problem by the ancients.[15]

The ancients solved the problem only with rectangular spherical triangles.[16] By drawing the altitude of the triangle in the summit C we can write, see figure 3:

On figure 2: A is the North Pole
B is the examined town T
C is Jerusalem J.
Hence: sin CD = sin AC * sin A
tang AD = tang AC * cos A. (figure 2 left)
tang AD = tang AC* cos (180°– A) = – tang AC * cos A. (figure 2 right)

2. In the triangle BDC, rectangular in D we know BD and CD; we can write
tang B = tang CD / sin BD

Example. T = New York =B $\lambda_T = 73.8°$ W $\varphi_T = 40.8°$ N
J = Jerusalem = C $\lambda_J = 35.2°$ E $\varphi_J = 31.8°$ N

sin CD = sin AC * sin A = sin 58.2° sin 109°. Hence CD = 53.47°
tang AD = – tang AC * cos A = –tang 58.2° cos 109°. Hence AD = 27.70°
BD = 27.70° + 49.2° = 76.90°

[13] See appendix 3: Rectangular spherical triangles.
[14] B. Bava Batra 25b.
[15] Those living before Delmedigo and those rabbis who rested on Sefer *Elim* until the end of the 18th century in east Europe.
[16] For the formulas of the rectangular spherical triangle, see the mathematical appendix 2: Rectangular spherical triangles.

$\Delta B = 54.19°$

This method is not fundamentally more difficult. But imagine the difficulty of calculation for people who could not use logarithms.
The rabbis and scholars of the sixteenth century, preceding the publication of *Sefer Elim* and the Canon Mathematicus, could use this method only if they had access to textbooks of mathematics written in Latin. The rare tables of trigonometric functions gave generally results in sexagesimal notation. Arithmetic operations like multiplication and division of trigonometric functions were then nearly inextricable.
Among all the rabbis of the history, only two of them made a practical calculation of the

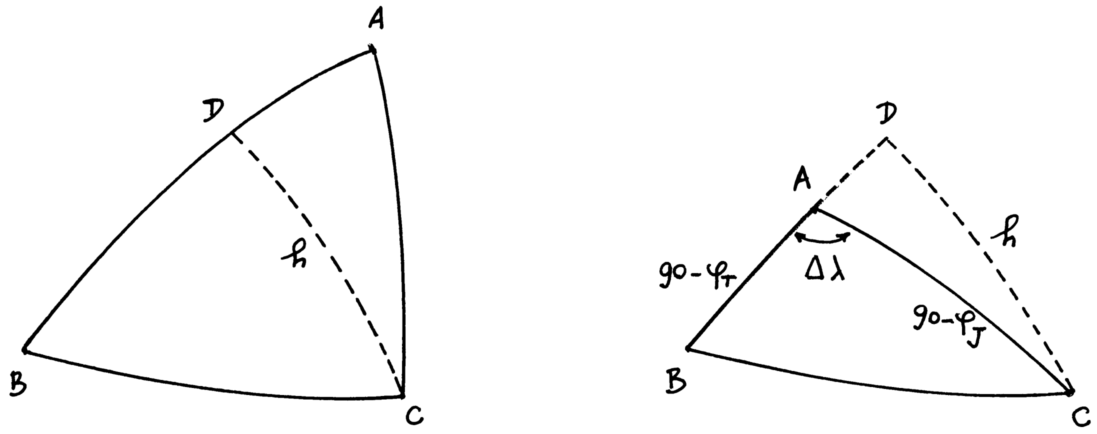

Figure 3: Solution of the ancients with only rectangular spherical triangles. We must plot the fitting altitude allowing the decomposition of the triangle into two rectangular spherical triangles for which we have sufficiently data. Right: angle A < 90°; left: angle A > 90°.

4. Locus of the places on the earth where the prayer direction is exactly eastward or westward.

latitude	$\Delta\lambda$ in degree	λ_W in degree.	λ_E in degree
32	7.14	28.06 E	42.34 E
34	23.19	12.01 E	58.39 E
36	31.42	3.78 E	66.62 E
40	42.36	7.16 W	77.56 E
44	50.05	14.85 W	85.25 E
48	56.05	20.86 W	91.26 E
52	61.03	25.83 W	96.23 E
56	65.28	30.08 W	100.48 E
60	69.02	33.82 W	104.22 E
70	76.96	41.76 W	112.16 E
80	83.72	48.52 W	118.92 E
85	86.89	51.69 W	122.09 E
90	90	54.80 W	125.20 E

Table 1: Locus of the places on the earth where the prayer direction is exactly eastward or westward.

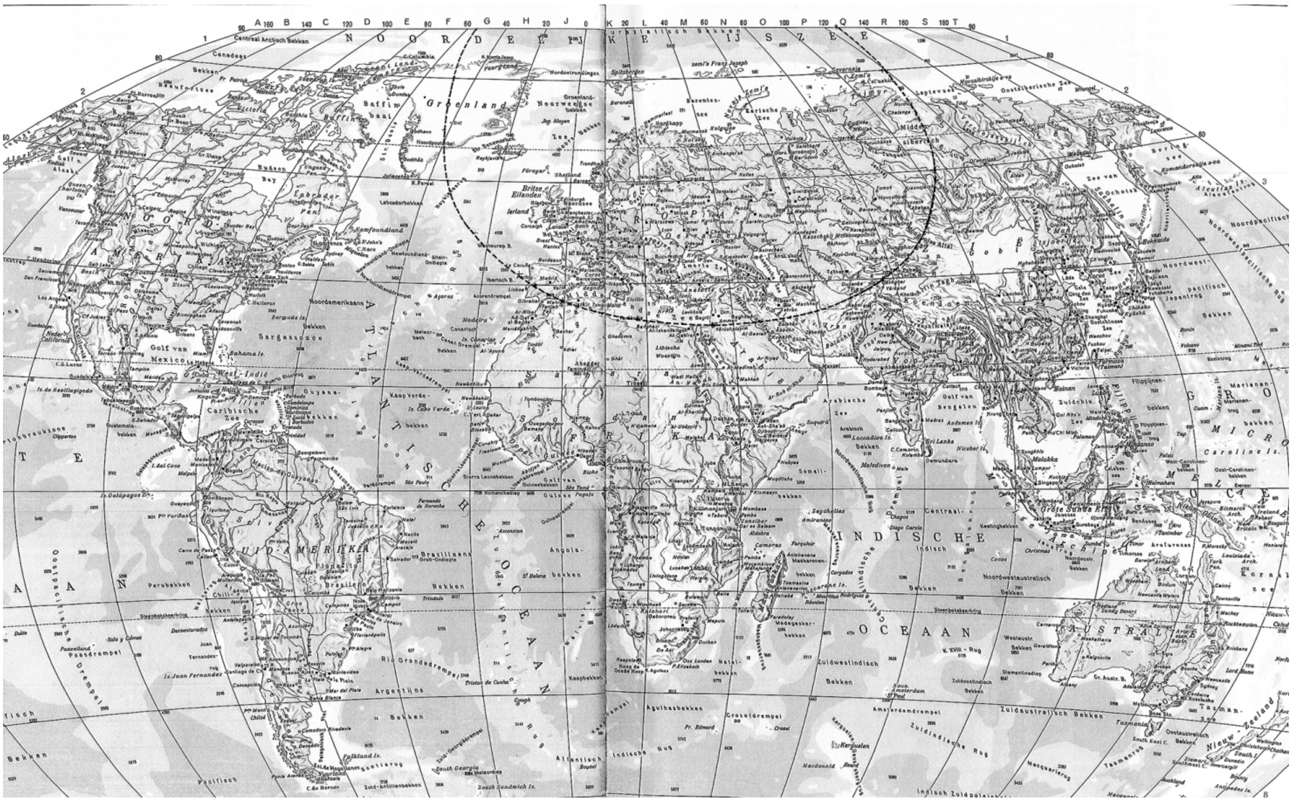

Figure 4: Locus of the places on the earth where the prayer direction is exactly eastward or westward. The locus passes through southern Spain (Andalusia) and not through southern France as R. Shneor Zalman wrote. Indeed, he worked with the coordinates given in *Sefer Elim*. Acknowledging this property proves that he mastered the subject.

prayer direction. R. Solomon Aviad Sar Shalom Basilea at the end of his book *Emunat Hakhamim*[17] and R. Shneour Zalman of Liady in p. 11 of his *siddur*.[18]

Let us consider the formula of the cotangents:[19]

$$\tan \varphi_J * \cos \varphi_T = \sin \varphi_T * \cos \Delta\lambda + \sin \Delta\lambda * \cot B.$$

If $B = 90°$ then: $\cos \Delta\lambda = \tan \varphi_J / \tan \varphi_T$.
Conversely, if $\cos \Delta\lambda = \tan \varphi_J / \tan \varphi_T$ then $B = 90°$.

Example.
If $\varphi_T = 40°$ we will have $B = 90°$ if $\cos \Delta\lambda = \tan \varphi_J / \tan \varphi_T = \tan 31.8° / \tan 40° = 0.74$ or if $\Delta\lambda = 42.36°$.
Thus $\lambda_T = 35.2 + 42.36 = 77.56°$ E the prayer direction is westward.
$\lambda_T = 35.2 - 42.36 = -7.16°$ W the prayer direction is eastward.

[17] 1730.
[18] Beginning of the 19th century.
[19] See appendix 2: Formulas of the spherical trigonometry..

5. Variation of the prayer direction angle with the north at different latitudes in function of the variable difference of longitude with regard to Jerusalem.

We choose three latitudes, see figure 5.

We see that for latitudes north of Jerusalem, the prayer direction evolves in function of $\Delta\lambda$ on the following way: B = 180° for $\Delta\lambda = 0$ (southward); then it diminishes regularly (southeast direction) until 0° for $\Delta\lambda = 180°$ (northward). Thus when $\Delta\lambda$ increases the southeast direction becomes for a sufficient $\Delta\lambda$ eastward and then for greater $\Delta\lambda$ it becomes northeast.

For latitudes south of Jerusalem, the prayer direction evolves in function of $\Delta\lambda$ on the following way: B = 0° for $\Delta\lambda = 0$ (northward); then it increases regularly until a maximum which remains less than 90° and then B diminishes again until B = 0° for $\Delta\lambda = 180°$ (northward). Thus when $\Delta\lambda$ increases the eastward prayer direction is never reached and the eastward prayer direction has always a northern component.

For the same latitude as Jerusalem, the prayer direction is always eastward with a slight northern component.

6. Allowable error on the prayer direction.

According to *Shulhan Arukh* and *Tur Orah Hayim* 94: 1:

When we are outside Israel we must pray toward Israel, in Israel we must face Jerusalem, in Jerusalem we must face the Temple and in the Temple we must face the Holy of Holy.

This ruling is based on B. Berakhot 30a and Tosefta Berakhot 3: 16.
In the commentary *Perisha*[20] on the *Tur* it gives the correct explanation of this passage. These different limits correspond to the different targets of diminishing sizes proposed to a thrower of arrows as a function of his distance to the target. Thus for someone standing outside of Israel the required precision depends on the distance to Israel i.e. $\Delta\lambda$ and it is determined by the requirement to reach Israel.

If we consider that the most northern point of Israel is Metula: λ= 35.6° E and φ = 33.3° N and the most southern point of Israel is Eilat λ= 34.95° E and φ = 29.56° N we calculate that in New York B_{min} = 52.86° and B_{max} = 55.97°.
We had found B = 54.19°
Thus B_{min} = 52.86° = 54.19° – 1.33°
B_{max} = 55.97° = 54.19° + 1.78°.

The required precision increases as the distance increases.

[20] R. Joshua ben Alexander ha-Kohen (~ 1555 – 1614)

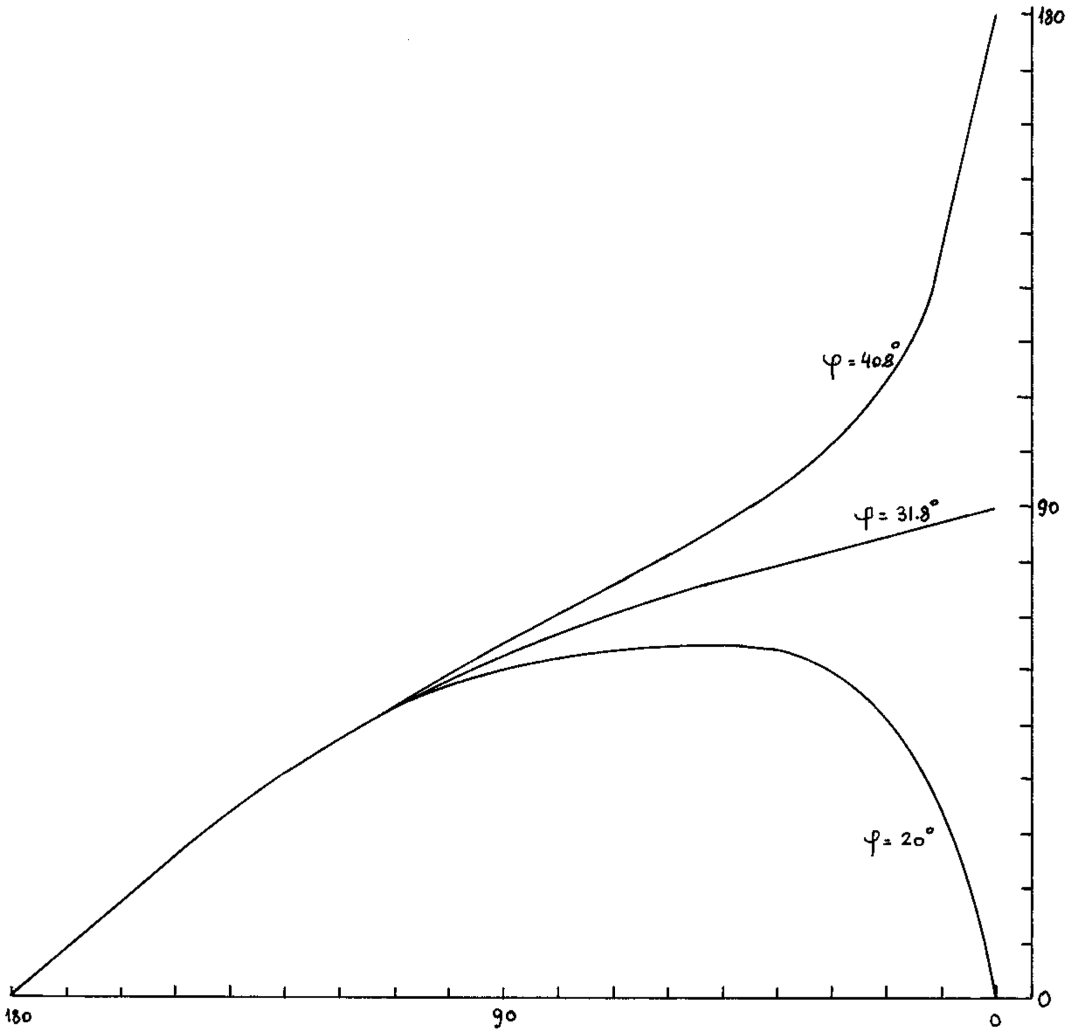

Figure 5. Prayer direction: angle between the north direction and the direction toward Jerusalem for towns situated to the west of Jerusalem, as a function of the difference of longitude for three different latitudes, φ = 40.8° (as New York), φ = 31.8° (as Jerusalem) and φ = 20° (similar to Mexico). *Levush* (R. Mordekhai Jaffe 1532 – 1612) had considered Central Europe i.e. a difference of longitude between about 10° and 30° till maximum 35°. This explains why he did not mention or apprehend that the prayer direction gets a northern component for a greater difference of longitude and why for the latitude of 31.8° he considered that the prayer direction is eastward, see more details further.

7. The state of geographical knowledge.

If we exclude the discovery of America and its tremendous consequences, the improving of the precise geographical knowledge of the world, the size of the continents and the geographical coordinates of the main towns in Europe and in the world was slow.
The representation of the ancient world at the beginning of the 16th century was still similar to the representation of Ptolemy and the ancients. We note that the ancients had generally a good notion of the latitude of the different places but their knowledge of the longitude of these places was imprecise. The correct determination of the longitude was solved only in the mid-eighteenth century when John Harrisson succeeded to build an accurate marine chronometer allowing the precise determination of the longitudes.

Therefore the longitude of the different towns of Europe remained inaccurate until the 18th century. This situation explains why we find so much inaccuracy in the geographical data of the rabbis as late as the end of the 18th century. Their unique source of information was the *Sefer Elim*. We find in this book at the end of *Hukot Shamayim*,[21] a table of the geographical coordinates of 102 locations.

Towns	*Sefer Elim*: Ancient Values			Modern Values		
	λ	φ	Δλ	λ	φ	Δλ
Jerusalem	66°		0°	35.22° E	31.78° N	0°
Bayonne	17°; 30’	42°; 50’	48.83°	1.47° W	43.50° N	36.68°
Brussels	26°; 42’	51°; 24’	39.30°	4.35° E	50.85° N	30.87°
Cordoba	9°; 40’	37°; 50’	56.33°	4.77° W	37.88° N	39.98°
Cracow	42°; 40’	50°; 12’	23.33°	19.92° E	50.05° N	15.30°
Lisbon	5°; 10’	39°; 38’	60.83°	9.13° W	38.73° N	44.35°
Lvov	43°; 15’	50°; 30’	22.75°	24 E	49.83° N	11.22°
Moscow	75°; 10’	61°; 15’	–9.17°	37.55° E	55.70° N	–2.33°
Prague	39°; 15’	50°; 10’	26.75°	14.43° E	50.10° N	20.78°
Tunis	33°	32°; 30’	33°	10.22° E	36.83° N	25°
Toledo	10°	40°	56°	4.03° W	39.87° N	39.25°
Vilnius	52°	53°; 30’	14°	25.32° E	54.67° N	9.90°

Table 2: Ancient and modern coordinates of different towns. We note that the values given by Delmedigo are still similar to those given by Ptolemy, Egypt (90 CE – 168 CE) in his Geography. Both considered that the longitude of Jerusalem is 66°. The origin of their longitudes was in the Canaries Islands. Δλ is the difference of longitude of the considered town with Jerusalem.

From this table we see the importance of the differences of longitude with regard to the reality. The ancients stretched Europe in the direction of longitude. The distance between Jerusalem and Lisbon was increased by more than 15° and the distance between Jerusalem and Moscow was increased by nearly 7°, so that Europe was stretched by about 22° in the direction of longitude.

8. A surprising different solution to the problem of the prayer direction.

This solution finds its origin in an erroneous understanding of the Mercator maps.
The end of the 15th and the 16th centuries were the period of the great successes of the Portuguese navy. Portugal was a major seafaring country. The Portuguese king prohibited the use of the “newly high-tech” globes for navigation, probably in order to prevent them from falling into foreign hand. In this period of the early 1500s navigators came to realize that a course of constant bearing is not the same as a great circle. The navigators came to realize that following the path of a great circle presents navigation handicaps and drawbacks. Indeed the sailor must be ever changing the compass direction with respect to those converging meridians if he wants to stick the oblique great-circle route. Thus the initial compass direction of a great-circle route will be incorrect as soon as the journey begins because an oblique great-circle direction, with respect to the north-south meridians, is different for every point. For practical reasons the sailors preferred to follow a course of constant bearing.

[21] pp. 289-290 in the Odessa edition.

The New Christian (he was converted as a child) Pedro Nunez[22] (~ 1502 – 1577) was appointed Royal Cosmographer in 1529 and Chief Royal Cosmographer in 1547. He wrote important works on the science of navigation, in Portuguese and later in Latin. He was the first to understand why a ship maintaining a steady course would not travel along a great circle, which is the shortest path between two points of the earth, but would instead follow a spiral course. In "De Arte Navigandi", Coimbra 1546, he announced his discovery and analysis of the curve of double curvature called the rumbus[23] or loxodrome. He showed that the orthodrome is the shortest distance between two points of the earth and not the loxodrome as many believed. It is the line traced by a ship cutting the meridians at a constant angle. We can also characterize this line as a spherical helix.
A major development in the construction of maps for the navigation was the construction in 1569 by Gerhard Kremer (1512 – 1594) of Rupelmonde, Belgium, Latinized into Gerhardus Mercator, of his world map. It was a great wall-map of the world on 18 separate sheets. It was entitled "*New and more complete representation of the terrestrial globe properly adapted for its use in navigation*". He was living at this period in Duisburg because of his problems with the inquisition.
This map was built according to the principle of the cylindrical projection from the rotation axis of the earth, of the sphere on a cylinder circumscribed to the earth along its equator. The cylinder was then cut along one of its meridian lines and then it was developed on a plane. In this map representation the parallels are horizontal lines and the meridians are vertical equidistant lines.
The Mercator projection had a great virtue that a straight line in the map is a rhumb on the globe of the earth and angles on the map equal angles on the earth. To set a course from one location to another, a navigator drew a straight line on the map and determined the bearing on it. The Mercator projection became the standard for navigations until modern times. It became also the standard for atlases, wall maps and geography books. But because of its distortions it was also the source of many errors of appreciation. See figures 6 and 7.
It was the origin of the wrong orientation of the mosques in North America because they determined the quibla (the direction of the prayer) according to the rhumb line.
Today mosques are built according to the *quibla* found by calculating the initial compass direction of the shortest distance to Mecca (thus the great circle route) using precise geographic coordinates. It is interesting to note that among Muslims, the direction of prayer (*quibla* in Arabic) was initially, as it is among Jews, toward Jerusalem. However, within two years of Muhammad's foundation of Islam (620-622), the Muslim *quibla* was changed from Jerusalem to Mecca. This was due perhaps in part to Muhammad's disappointment that few Jews were converted to Islam. The Muslims were then instructed to face the direction of Mecca. Thus this requirement that the Muslims follow rigorously, much more rigorously than the Jews, is the consequence of the desire to please the Jews. It proves that in the beginning of the seventh century the Jews of Medina followed the ruling of the Talmud B. Berakhot 30a and prayed toward Jerusalem.
It is certain that Jews were also influenced by the use of the maps constructed according to the Mercator projections and thought incorrectly that the rhumb line (see figure 7 and 9) was the shortest distance between two points of the earth and the correct manner to face Jerusalem. But this influence was much less however than the Muslims, because Jews consider this obligation as an à priori obligation only. A posteriori they accept other orientations. They accept the consequences of town planning regulations. When we examine the orientation of the modern and ancient

[22] According to some references his birthday would be somewhere between 1492 and 1502. A birth in 1492 makes sense; it would be strange that he was converted later after 1492.
[23] A word of old French of Latin origin: rumbus.

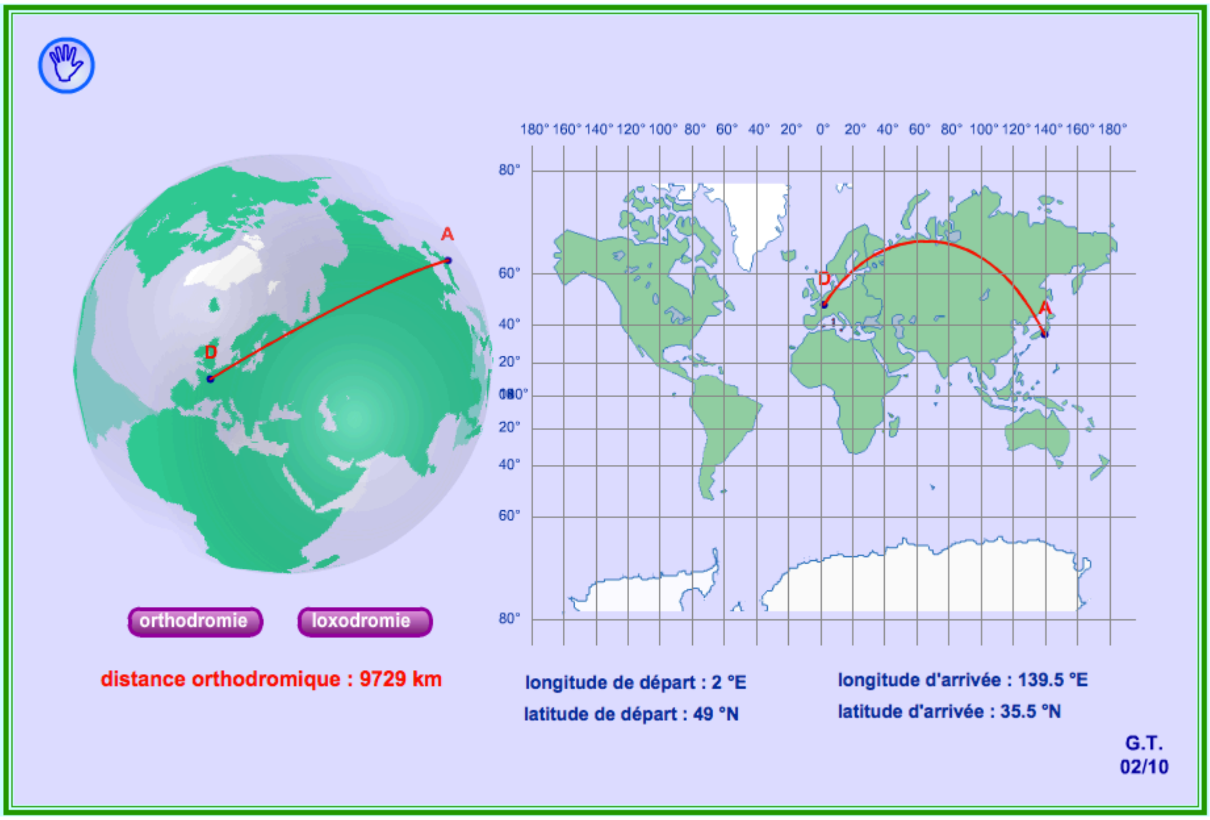

Figure 6 : The geodesic line or orthonome joining D to A, with left a perspective view and right the representation on a Mercator map. On this figure D is Paris and A is Tokio. This figure could be easily adapted and interpreted if we consider that D is Denver Colorado and A is Jerusalem. The curves would be similar.

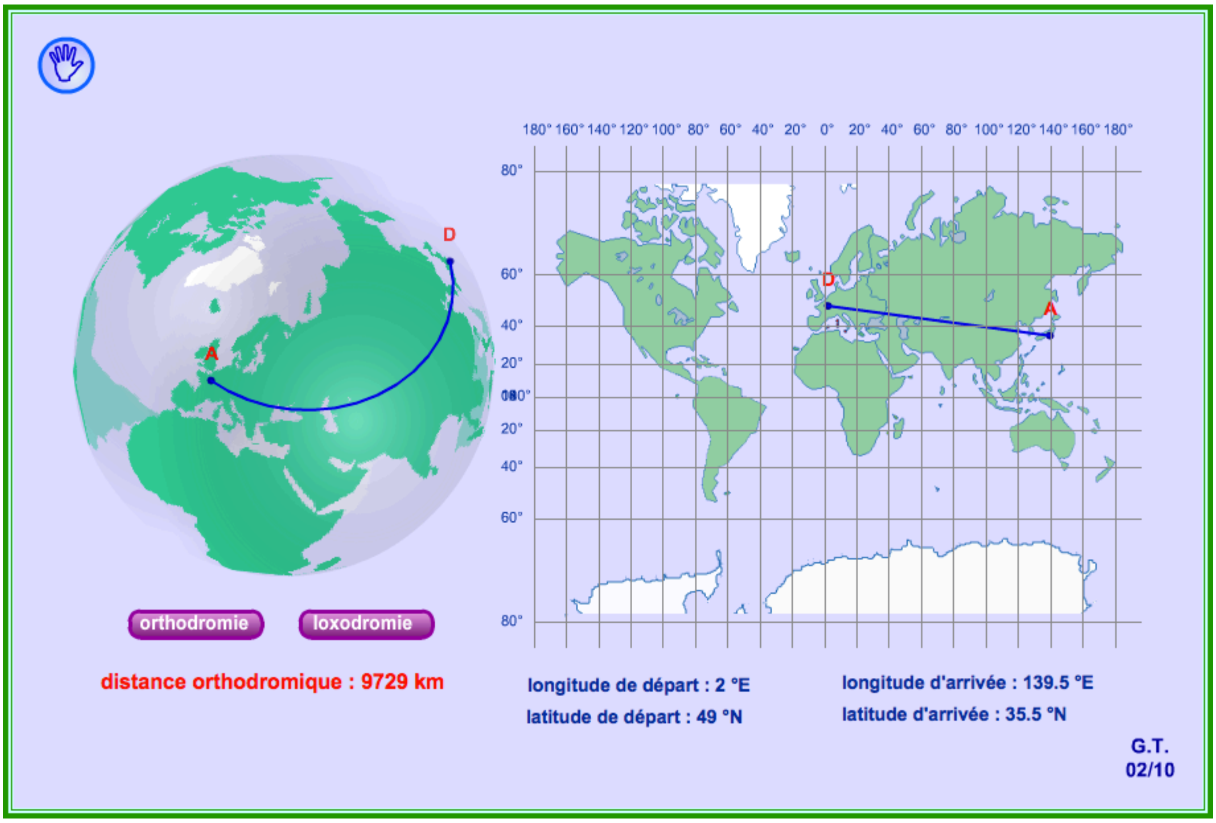

Figure 7 : The rhumb line or loxodrome joining D to A.

synagogues we ascertain that generally their orientation was incorrect. This was also the case of the ancient synagogues of Central Europe. In America the orientation of most of the synagogues results from the disposition of the streets and the avenues of the different towns. Today the importance of the rhumb lines results also from a new trend in *Halakha* in their favor. We will come back to the subject after we will examine the different halakhic opinions about the prayer direction. Indeed the plain reading of the *Levush* and *Mishnah Berurah* can give the impression that these texts were written on the basis of an incorrect reasoning made on a Mercator map.

The mathematical problems connected with the rhumb lines remained unsolved; the determination of the bearing must be calculated graphically on the Mercator map. The invention of the logarithms (1614) and of calculus (Newton and Leibnitz around 1684 – 1687) allowed Leibnitz to establish the equations of the loxodrome at the beginning of the 18th century. The theoretical solution of the problem is given in the Mathematical Appendix 4. The practical solution is given by the following formulas:[24]

$$\lambda_2 - \lambda_1 = \text{tang}\ \alpha\ [\text{Ln tang}\ (\pi / 4 + \varphi_2 / 2) - \text{Ln tang}\ (\pi / 4 + \varphi_1 / 2)]$$

$$s = (\varphi_2 - \varphi_1)\ R / \cos \alpha$$

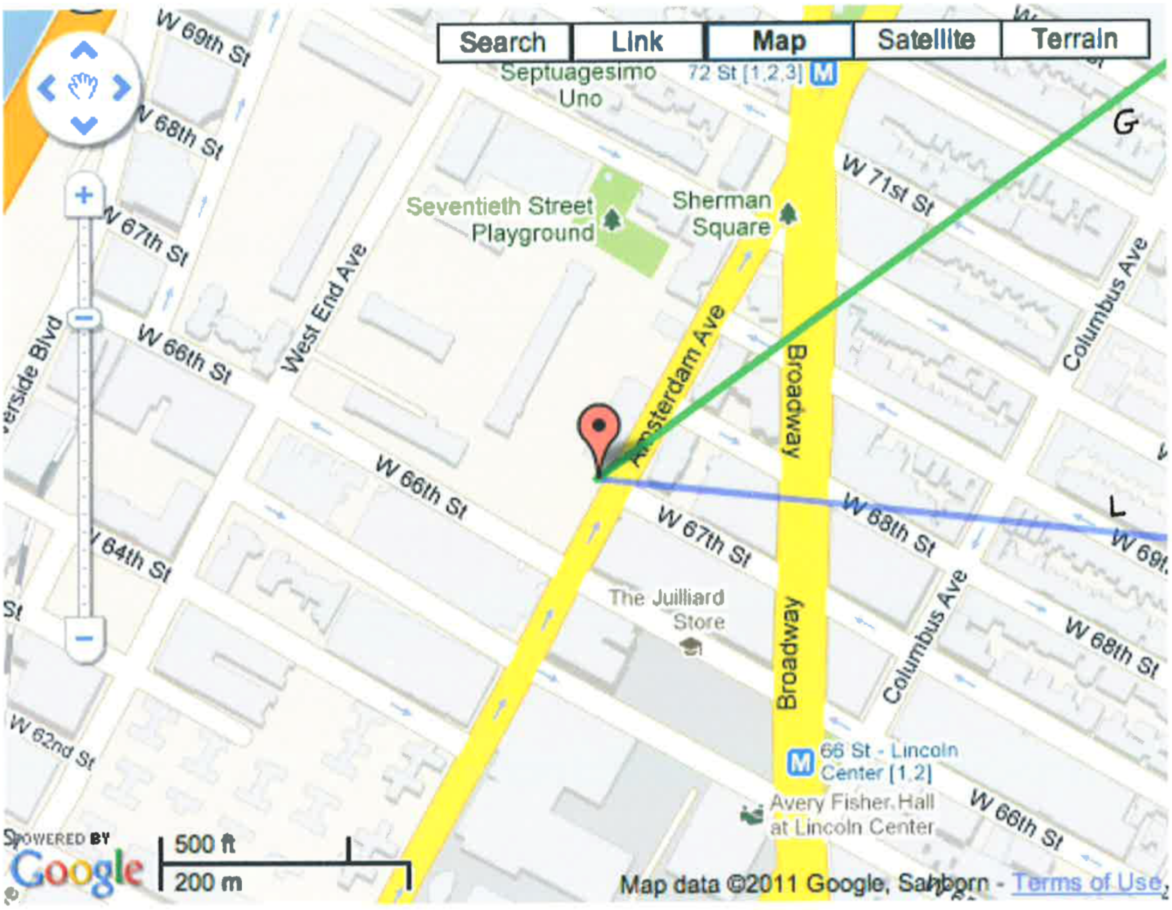

Figure 8: The Prayer direction in Manhattan NY according to the great circle theory: straight line G with the bearing angle B = 54.01° and according to the rhumb line theory: straight line L with the bearing angle α = 95.8°. The considered location is on Amsterdam Avenue, at the location of the new Lincoln square synagogue, allowing the comparison between the theoretical and the practical prayer direction.

[24] See mathematical appendix 4: Orthonome and loxodrome.

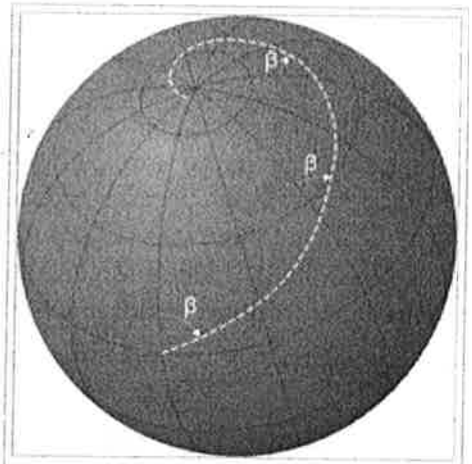

Image of a loxodrome, or rhumb-line, spiraling towards the North Pole

Figure 9: Perspective representation of a rhumb line joining two points β. The bearing of this rhumb line is about 45°.

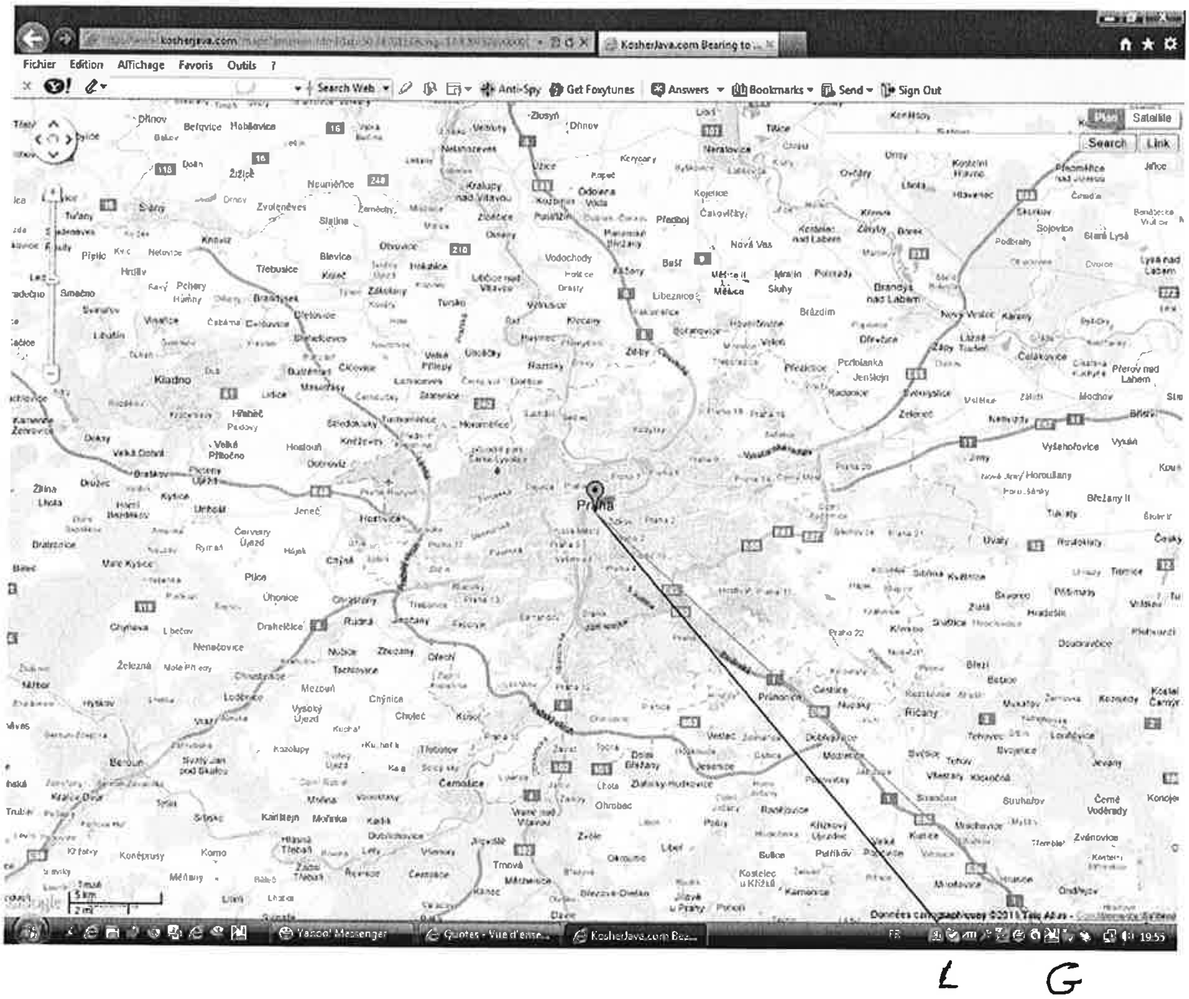

Figure 10: The prayer direction in Prague according to the great circle theory: straight line G with the bearing angle B = 132° and according to the rhumb line theory: straight line L with the bearing angle α = 139.66°.

Example 1: T = New York $\lambda_T = 73.8°$ W $\varphi_T = 40.8°$ N
J = Jerusalem $\lambda_J = 35.2°$ E $\varphi_J = 31.8°$ N

Tang α [0.781244 – 0.585921] = – 109 * 2π / 360
α = 95.86° where α is the angle of the prayer direction with the northern direction.

$s = (\varphi_2 - \varphi_1) R / \cos \alpha = (40.8 - 31.8)*(6371.221 / \cos 95.86°) * (2\pi /360) = 9802.23$ km.

Example 2 : T = Prague $\lambda_T = 14.4°$ E $\varphi_T = 50.1°$ N
J = Jerusalem $\lambda_J = 35.2°$ E $\varphi_J = 31.8°$ N

Tang α [1.0134012– 0.585921] = – 20.80 * 2π / 360

α = 139.66°

Modern Coordinates[25]		
Town	Great circle	Rhumb line
	B= Angle with the meridian	α = Angle with the meridian
Bayonne	B = 99.63°	α = 112.02°
Lisbon	B = 86.97°	α = 100.84°
Lvov	B = 150.89°	α = 155.08°
New York	B = 54.01°	α = 95.80°
Prague	B = 132°	α = 139.66°
Toledo	B = 91.81°	α = 104.26°
Tunis	B = 88.41°	α = 95.58°
Vilnius	B = 158.74°	α = 162.74°

Tabel 3: Recapitulative table of prayer directions. The prayer direction in some towns according to the theory of the great circle and according to the theory of the rhumb line.

Ancient Coordinates (*Sefer Elim*)		
Town	Great circle	Rhumb line
	B= Angle with the meridian	α = Angle with the meridian
Bayonne	B = 89.45°	α = 105.91°
Lisbon	B = 79.20°	α = 99.03°
Lvov	B = 129.36°	α = 137.83°
Prague	B = 122.67°	α = 132.60°
Toledo	B = 82.07°	α = 100.26°
Tunis	B = 82.44°	α = 91.44°
Vilnius	B = 149.53°	α = 154.98°

Table 4: Recapitulative table of prayer directions. The prayer direction in some towns according to the theory of the great circle and according to the theory of the rhumb line.

[25] The numbers mentioned in the table, in the right column, correspond to the results given on the website Kosher Java. They differ slightly from those calculated in this paper. There are several reasons for the slight differences: the precision of the coordinates, the size of the town (for example New York!) and the taking into account of the real shape of the earth. In the case of New York, we found above B = 54.19° and α = 95.86°

9. Special Situations. Additional drawbacks of the second solution.

In Alaska at the longitude 144.8°, the difference of longitude with regard to Jerusalem is 180°. The prayer direction according to the theory of the great circle, is northward, along the tangent to the meridian.
If we use the variant solution of the rhumb line there is an indetermination: we have two solutions, southwestward and southeastward. The two rhumb lines joining the chosen location to Jerusalem are equivalent and have the same length. Now if we consider two neighboring locations, the one slightly to the east of this meridian and the second slightly to the west of this meridian, they will have two quasi-opposite prayer directions, southeastward and southwestward.
If we consider the particular location $\lambda = 144.8°$ and $\varphi = 65°$.

Great circle solution: The prayer direction is northward. The distance to Jerusalem is:
$(25 + 58.2) * 2\pi * 6371.221 / 360 = 9251.74$ km.

Second solution: rhumb line.
We find $\alpha = 73.67°$ or $\alpha = -73.67°$.
The length of these two rhumb lines is: $s = (65 - 31.8)\ 6371.221 * 2\pi / (\cos 73.67° * 360) = 13129$ km. The length of the two rhumb lines is about 142% of the length of the great circle.

10. Halakhic survey.

In the Talmud there are different opinions about the prayer direction (B. Berakhot 30a and B. Bava Batra 25a and b)

a. The prayer direction is toward Jerusalem. More exactly it is toward Israel when we are outside of Israel, it is toward Jerusalem in Israel and it is toward the Temple in Jerusalem and finally it is toward the Holy of the Holy in the Temple.

b. The Providence is everywhere, so is the prayer direction (except for eastwards according to Rav Sheshet because of the worship of the Christians and the idolaters).[26]

c. The Providence is westward and the prayer direction is westward.[27]

All the rulers followed the first opinion. However in the case of a traveler who has no orientation or in the case of the impossibility to orient correctly the synagogue, the rulers rest à posteriori on the second opinion.

Tossafot[28] wrote that we are living in the west and therefore our prayer direction is the east. We have no information about their geographical knowledge; it is even possible that they considered a flat earth.

R. Asher ben Jehiel (~ 1250 – 1327) and R. Jacob ben Asher (~1270 – 1340) in *Tur* ruled explicitly that the prayer direction is toward Israel and Jerusalem in agreement also with the

[26] This opinion was followed by R. Isaiah ben Elijah de Trani in *Shiltei ha-Giborim* and is in agreement with the fact that the remains of the ancients churches were oriented eastwards.
[27] The exact meaning of the third opinion remains unclear: does it mean the west of the Temple or the west in the absolute?
[28] B. Berakhot 30a: בד"ה לתלפיות. The explanation given by R. Shneor Zalman and relating this opinion of Tossafot to the fact that the prayer direction in southern France is eastward when we use the ancient coordinates of Sefer Elim or of the Geography Ptolemy (90 – 168) seems unlikely.

final dictum in B. Bava Batra 25b that the Babylonians must pray in a southwest direction. This ruling contradicts the other opinions mentioned there. He recalls the statement of Tossafot that they pray eastward. But in Toledo this statement appears now to be correct.

In the ***Shiltei ha-Giborim*** on the hilkhot ha-Rif[29] it mentions the ruling of R. Isaiah ben Elijah of Trani (the younger) known as Riaz[30] according which we pray in any direction if one cannot orient oneself except for eastward because of the idolaters.[31] He writes further that their synagogues were oriented toward the southeast.

R. Moses Isserles (~ 1525-1530, –1572) included this ruling in his glosses[32] on Shulhan Arukh Orah Hayim 94. 2 and in Darkei Moshe on Tur Orah Hayim 94.[33]

The gloss of Rema can be misunderstood and seems even contradictory.

הגה: ואנו שמחזירין פנינו למזרח מפני שאנו יושבים במערבה של א"י ונמצא פנינו לא"י (טור וסמג).
אין עושין מקום הארון וצד התפילה נגד זריחת השמש ממש כי זהו דרך המינים רק מכוונים נגד אמצע היום (הגהות אלפסי החדשים)[34]
ומי שרוצה לקיים אמרם: הרוצה להעשיר יצפין או להחכים ידרים, מכל מקום יצדד פניו למזרח.

The text must be divided in three parts.
The first sentence is a quotation of *Tur, Semag*[35] and *Tossafot*[36] according which we pray eastward. *Rema* seems to change slightly the meaning of this quotation; otherwise it would contradict the second sentence. This first sentence is related to the preceding statement of *Shulhan Arukh* about a man praying in another direction because he is riding or because the synagogue is not correctly oriented (for example due to government regulations). In these cases the worshipper must at least turn his face eastward.
The second sentence of the gloss is related to the position of the Ark and the "east wall" of the synagogue. It must not be perpendicular to the east direction but perpendicular to a direction deviated southward.
The third sentence of the gloss is related to those people who want to attain another object and want to stand toward the north in order to become rich or toward the south in order to become clever. They should also turn their face eastward.
In fact the third sentence could have been gathered to the first. Anyhow these statements don't seem contradictory. In the first and the third statement the worshipper must turn his face

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[29]R. Joshua ben Simon Barukh (end of the 16th and first half of the 17th century) was an important rabbinical leader whose activity was connected to the development of the printing. He published *Shiltei ha-Giborim* on the *hilkhot ha-Rif* and on the *Mordekhai* of R. Mordekhai ben Hillel in Sabionetta (1554-1555). In the *Shiltei ha-Giborim* on the *Hilkhot ha-Rif* he quoted extensively the rulings of R. Isaiah ben Elijah di Trani (lived at the end of the 13th century) , the grandson of R. Isaiah ben Mali di Trani and he maintained them alive. The importance of this scholar has been forgotten. However any Talmudic student uses his *Ein Mishpat* and *Ner Mitsva* and rests on his erudition.
[30] ריא"ז: רבי ישעיה אחרון ז"ל.
[31] The ruling of Rav Sheshet.
[32] Critical and additional notes on *Shulhan Arukh*: the *Mappah* was published in Cracow in 1569-1571.
[33] Commentary on Tur published in two versions: the first was the long version, the first part on Orah Hayim was published in Fuerth (1760); the second version was an abridged version by the author and it was published in Berlin (1702-1703).
[34] These glosses were written by R. Joshua ben Simon Barukh, see note 29.
[35] Sefer Mitsvot Gadol by R. Moshe of Coucy (13th century), grandson of R. Hayim Cohen of Paris, the most reputated pupil of R. Tam.
[36] See note 28 above.

eastward but he is not supposed to be able to orient his face precisely to the southeast direction. By contrast in the second statement we are during the building of the synagogue and the builders must be able to measure precisely the orientation and the implantation of the synagogue.[37]

R. Mordekhai Jaffe (1535-1612) was the pupil of R. Moses Isserles and Solomon Luria (~1510 – 1574). He left Bohemia in 1561 for Italy where he stayed 10 years. He came back in about 1571 and was appointed head of the Yeshiva of Grodno in Poland. In 1592 he became *av beit din* in Prague, in succession of R. Judah Loeb ben Betsalel when the latter was appointed to Posen. In 1599 he switched posts with R. Judah Loeb ben Betsalel who returned to Prague. R. Jaffe remained in Posen until his death.
In *Levush ha-Tekhelet* on *Orah Hayim* 94. 3 R. Jaffe follows the ruling of *Shulhan Arukh* and the precisions of Rema in his gloss but he adds a paragraph explaining the gloss of Rema.

ודע שכל הארצות האלו אשר אנו מפוזרים בהם כולם הם כנגד מערבית צפונית של ארץ ישראל, ואינם מכוונים במערבה של ארץ ישראל ממש, ואריכות הימים והלילות יוכיחו זה למי שיודע מעט בצורת הכדור, לפיכך נראה לי שטוב ונכון הוא כשעושין בית הכנסת שיזהרו שיעשו הכותל המזרחי שעושין שם הארון ומתפללים כנגדו שתהא נוטה קצת לצד מזרחית דרומית, ואז נעמוד מכוון כנגד ארץ ישראל וירושלים ובית המקדש וקדשי הקדשים, וגם לא נחקה המינים שיאמרו שאנו מתפללים כנגד השמש כמותם.

אלא שזה צריך עיון באיזה אופן נעמידנה באילו הארצות, **ולפום ריהטא היה נראה לי** על פי תמונת צורת הכדור שאם נעמידנה באופן שביום תקופת ניסן או תקופת תשרי, או סמוך להן בששה או בשבעה ימים, כשתזרח השמש בבקר ותכנס בחלון שבאמצע כותל המזרח מן הבית הכנסת ויכה הניצוץ כנגדה על הכותל המערבי נוטה הניצוץ מן אמצע הכותל המערבי לצד דרום רחוק מן האמצע באופן שבכמו חצי שעה או שעה אחר הזריחה תגיע אל אמצע הכותל ממש מול חלון המזרחי, בית הכנסת כזו ודאי היא עומדת ממש באלו הארצות נגד ירושלים ובית המקדש לפי תמונת הכדור. אבל אם נעמידנה באופן כשתזרח החמה בימים הנ"ל בחלון המזרח ויכה ניצוץ השמש המערבי ממש כנגד החלון באמצעו מיד בעת הזריחה, זהו ממש כנגד המזרח, וזהו חק המינים לעשות כן, ואם היינו עושים כן היינו מחקים את המינים, כלומר מחזיקים דבריהם וחקם, וגם אין משתחוים נגד ירושלים ובית המקדש רק בצדם, לפיכך לא נעמוד הבית הכנסת באופן זה. ואם נעמידנה באופן שבעת הזריחה בימים הנ"ל תכה השמש בכותל המערבי נוטה לצד צפון, אף על גב שגם בבית הכנסת כזו אינו מחקה המינים, מכל מקום אינה כתיקונה לפי פסוק והתפללו וגו' דרך ארצם, שזו אינה נוטה לא לירושלים ולא לבית קדשי בקדשים, אדרבא היא פונה פניה סתם לפי תמונה זו, והיא התמונה השלישית אשר ציירתי.

Thus R. Jaffe writes: “The countries of our dispersion (probably mainly Poland, Germany bohemia, Moravia and Italy) are northwestern of Israel and they are not at the same latitude. Therefore it seems that the prayer direction and the synagogue orientation should be southeastward”. He adds: “*at the first glance*[38] *it seems to me that* the prayer direction is the direction of the sun a half hour or an hour after sunrise on the day of the equinox or six, seven days later”.

During 140 years this additional and explanatory remark did not raise any objection.
R. Joshua ben Alexander ha-Kohen (~1555 – 1614) was a pupil of both *Rema* and *Maharshal*.
In his commentary on *Tur, Perisha*, he quoted the gloss of *Rema* and the complete quotation of R. Isaiah ben Elijah from *Darkei Moshe* and finally he added that in *Levush* the subject is explained a little more deeply.

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[37] The apparent contradiction between the second statement and the two other statements was not raised by any commentator.
[38] Free translation of : לפום ריהטא.

R. Joel Sirkes (1561 – 1640) added in his commentary *Bayit Hadash* after quoting *Levush:* "and all his words are in fact included in the words of R. Isaiah the Younger that we turn [also] southward as it is also written in the glosses [of Rema] on Shulhan Arukh"
R. Yom Tov Lippman Heller (1579 – 1654) in his commentary on Rosh copied the text of *Levush* and noted that, as he can ascertain, the requirements of *Levush* are not respected. He personally turned southeastward when sitting left to the Ark. By contrast if he was sitting right to the Ark he would not dare giving the impression to turn aside from the Ark.

We see that these three important authorities considered that the opinion of *Levush* did not differ from that of R. Isaiah the younger and *Rema*. For them the additional explanation of *Levush* did not add or change anything to the ruling of *Rema*.[39] They were probably not acquainted with the new map of Mercator and they were not struck by an anomaly in the explanation of Levush. They probably understood the subject according of the principle of the great circle as certainly did *Rema*.

In fact, the additional explanation of *Levush* is problematic. Indeed in the first paragraph there are two geometrical difficulties: first *Levush* justifies the southeastward prayer direction by the fact that these areas are located northwest of Israel. But he did neglect the influence of the difference of longitude between the examined location and Jerusalem. When this difference of longitude increases the prayer direction becomes northeastward. Second the text implies that for a location west to Israel with the same latitude the prayer direction would be eastward. In fact this is not true; the prayer direction in this case is slightly deviated to the north. However we can argue in the defense of *Levush* that he had only in mind the countries of Central Europe which are sufficiently near to Israel (small $\Delta\lambda$) and where the prayer direction is indeed southeastward. Similarly in the same area of $\Delta\lambda$, the deviation of the prayer direction with regard to the east direction is small and can be neglected. In other words the explanation of *Levush* was a simplistic and subjective explanation, but Levush did not adopt a new position, different from his predecessors. Anyhow the explanation of *Levush* did not raise objections until the beginning of the 18th century.

A point of the commentary of *Levush* was not yet exploited. *Levush* told us that in his area the prayer direction is given by the direction of the sun an hour after sunrise on the day of the equinox.[40]
On the day of the equinox, one hour after sunrise the hour angle[41] of the sun – 75°.[42]
In Prague we have $\lambda = 14.4°$ E and $\varphi = 50.1°$N
The zenithal distance of the sun is given by[43]
$\cos z = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos H = \sin 50.1° * \sin 0° + \cos 50.1° * \cos 0° * \cos(-75°)$
$= 0.1660$ and $z = 80.4435°$

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[39] It is then with great surprise that I ascertained that R. Moses Sofer wrote in *Hatam Sofer Orah Hayim* n° 19 about the *hiddush* of *Levush* about the southeast prayer direction. We have a similar surprise in responsum 80 where *Hatam Sofer* enumerates the different opinions about the entrance of Sabbath and ignores the position of "the *geonim*" also championed by Rambam.
[40] We consider an hour after sunrise on the day of the equinox and neglect the variants given by a half hour or 6 – 7 days after the equinox, because this choice will give us the maximum deviation with regard to the east direction. As we will see this deviation is very small.
[41] The hour angle is the distance measured on the equator of the circle of declination passing through the celestial body and the superior point of the equator..
[42] At the equinox the sun is on the equator. At sunrise the hour angle is thus –90° and an hour later it is –75°. At noon the sun coincides with the superior point of the equator and the hour angle is 0°.
[43] For a justification of the formulas of transformation of coordinates used on this page see: Astronomie Générale, Bakouline, Kononovitch and Moroz, Moscou 1974, pp. 62 – 63. See also Astronomical Algorithms, Meeus, J. Willmann-Bell, Richmond Virginia, 1991, pp. 88-89.

We can then calculate the azimuth of the sun:

$$\sin \delta = \sin \varphi \cos z - \cos \varphi \sin z \cos A$$

Now $\delta = 0$ and therefore $\cos A = \text{tang}\ \varphi\ \text{cotg}\ z$ with $\varphi = 50.1°$ and $z = 80.4435°$.
Hence $A = -78.38°$.

We can use another formula: $\sin A = \cos \delta \sin H / \sin z$.
Now $\delta = 0$ and therefore: $\sin A = \sin H / \sin z = \sin(-75°) / \sin(80.4435°)$.
Hence $A = -78.38°$.
The azimuth is measured from the south but the prayer direction B is measured from the north. Thus B = 101.62°. In fact this angle B is the maximum angle. *Levush* gave two limits and said in fact that B is between about 95.8° to 101.62°.[44]

With the coordinates of Prague according to the ancients this result is practically unchanged. We have indeed $\lambda = 39.25°$ and $\varphi = 50.1666°$.

This value of B = 101.62° must be compared with B = 122.67° according to the great circle theory and $\alpha = 132.60°$ according to the rhumb line theory. These two values correspond to the coordinates of Prague according to the knowledge of the ancients.
The difference between the direction adopted by *Levush* and the direction calculated is considerable and requires an explanation.
We must conclude that *Levush* probably was not able to make the trigonometric calculation of the prayer direction. This is a disappointing conclusion. The numerical indication that he gave for his prayer direction was thus determined either by the indications of a map or on the basis of a purely subjective impression.
Now we indeed know that the maps available at that time were inaccurate and even erroneous. However the main reason of their imprecision was the longitudes, while the latitudes were known with a good precision. This makes it impossible that the angle adopted by *Levush* would have been measured on a map. I had considered particularly the possibility that R. Jaffe got already acquainted with the new wall map that Mercator printed in 1569. R. Jaffe would have been the first to be mistaken by the distortion of the Mercator map. He would have measured the angle of the direction on the map and considered that the straight line of the map is the shortest distance between Prague and Jerusalem.
But the importance of the difference between the value adopted by *Levush* i.e. angle B between 95.8° and 101.62° and the value of about $\alpha = 132.60°$ that he should have measured on the map of Mercator excludes this possibility. The only possible and disappointing conclusion is thus that *Levush* adopted his prayer direction on estimation and subjective basis. Hence the rather broad interval admitted by him, i.e. that angle B may be included between 95.8° and 101.62°. We understand now better why he wrote at the beginning of the paragraph defining the prayer direction adopted practically by him: **ולפום ריהטא היה נראה לי**. It confirms us that this was not calculated but it was rather a subjective estimation. It also confirms that it would be rash to interpret the text of *Levush* literally and to ascribe him a new exegesis contradictory to his predecessors.

R. Yoseph Solomon Delmedigo (1591-1655) wrote, incidentally, in Mayan Hatum, a part of *Sefer Elim*:[45]

[44] In *Beour halakha*, the second commentary of *Mishna Berura*, the author recopies the practical rule of *Levush*. If we transpose the present calculations to Vilnius with $\varphi = 54.67°$, we find: $z = 81.3921°$, $A = -77.67°$ and $B = 102.33°$.
[45] Amsterdam 1628. The book was edited by R. Manasseh ben Israel.

ומכאן תבין שמצב ההיכל שבבתי כנסיות אינו מכוון נגד ירושלים יפה ואינם מתפללים דרך ארצם שלדרים בצפון והם מערביים לא יפה עושים אותו נגד מזרח. ובעלי מקרא מדקדקים בזה מאד כנראה בספר האדרת להחכם אליהו בשיצי.[46] והוא ותלמידו החכם כלב אפינדרפולו[47] היו תוכנים תלמידי מהר"ר מרדכי כומטיאנו[48] איש כלול בכל חכמה כנראה בכמה חבורים שכתב למודיים וטבעים, גם כל ספרי הראב"ע פירש.[49]

Thus the synagogues in Europe are generally incorrectly oriented. They are oriented eastward and this is not toward Jerusalem and their land. The Karaïtes are very strict on this issue as it appears from the book of their first mentioned leader.[50]

Tossefot Yom Tov on Mishnah Berakhot I, 1 wrote about Delmedigo in the most over polite terms:
מצאתי לרופא מומחה וחכם כולל מהר"ר יוסף שלמה דלמדיגא מן קנדיאה בספרו, בחלק שממנו שקראו בשם גבורת

Note that Demeldigo did not perform any calculation of the prayer direction. It would have been a good application of his theory of the rectangular spherical triangles. He calculated however the distance between two towns of the earth.

R. Solomon Aviad Sar Shalom Basilea (1680 – 1749)[51] had an extensive education in mathematics and astronomy.[52] He addressed the issue of the prayer direction in his book *Emunat Hakhamim*.[53]

He was actually the first author to take exception to R. Joffe's reasoning. He did not object the ruling of *Rema* but the erroneous reasoning of *Levush*. R. Jaffe had considered the sign of difference of latitude between the considered town and Jerusalem but he had neglected the effect of the difference of longitude. R. Basilea proved that in a town like Lisbon, the prayer direction is eastwards with a slight deviation to the north and not to the south, although the latitude of Lisbon is greater than that of Jerusalem.
R. Basilea presented in an appendix, written in Italian, a complete calculation of the prayer direction for Lisbon in order to give the necessary tool to anyone to perform correctly this calculation. The calculation was performed in a modern way, using the analogies of Napier and the logarithms (1614). The only remark is the imprecision of the longitudes adopted by R.

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[46] Elijah Bashyazi , Andrinople-Constantinople, 1420-1490.
[47] Caleb Afendopolo, Andrinople-Constantinople, second half of the 15th century.
[48] R. Mordehai Comitiano (1420- ~ 1487). His most important pupil was R. Elijah Mizrahi (~ 1450-1526).
[49] P. 435 in the Odessa edition, 1864.
[50] I could consult recently the little book שימת עין מאת חיים יחיאל בארנשטיין, ורשה תרע"ג, thanks to Rabbi Samuel Pinson of Brussels. Borenstein saw the book of Bashiazi ('ענין תפלה פרק ג) and he noted that his calculations were primitive. He assimilated spherical triangles to planar triangles.
[51] He was together with R. Isaac Lampronti (Pahad Yitshak) and R. Samson Morpurgo (Shemesh Tsedaka) considered as the important Italian rabbis at the beginning of the 18th century. In 1733, R. Basilea was at the center of a forgotten incident that Jews should never forget. As he was making his regular visit to prison of Mantua on a Friday afternoon, he bent over to put some money in the alms box as he was used, a Christian hooligan painted a cross on his rear. As he left the prison he was mocked by the host. He retorted: "You should not laugh if you notice where the cross has been placed". His response so infuriated the Church authorities that he was thrown into prison and held for almost a year despite his poor health. Even after his release he remained under house arrest until 1739 and the Chief Rabbi of Mantua was restricted to the ghetto until his death (Simonsohn, p. 158, History of the Jews in the Duchy of Mantua, Jerusalem 1977 and Ruderman p. 227, Jewish Thought and Scientific Discovery in Early Modern Europe, Detroit 1995).
[52] By contrast to the German and Polish rabbis of his time who in the best case had a partial, marginal and unavowable mathematical knowledge.
[53] Chapter 24, page 46b in the edition of the book in Yohannisburg,1859. The first edition was in Mantua, 1730. However, the Public Library of New York restricts the access to the old editions when there are more recent editions. The xerox copy of the calculation was not allowed.

Basilea: 9° 10' and 39° 38'for the longitude and the latitude Lisbon, 63° 30' and 32° for the longitude and the latitude of Jerusalem and hence a difference of Longitude of 54° 20'. These values are compatible and only slightly better than the values of *Sefer Elim* (1629) and the *Geographia* of Ptolemy (2nd century). R. Basilea found a prayer direction eastwards slightly deviated to the north, making an angle of 82° 29' (82° 20' after re-computation because of an imprecision at the end of the calculation) instead of 87° when the calculation is performed with the modern ability.
Because of its historical interest, we will present the mathematical solution of R. Basilea. His solution is based on the use of the two first analogies of Napier combined with the use of logarithms. R. Basilea surpassed certainly his contemporary and future colleagues by his mathematical knowledge and capacity.

$$tang\ \frac{B+C}{2} = \frac{\cos\frac{b-c}{2}}{\cos\frac{b+c}{2}}\ cotg\ \frac{A}{2}$$

$$tang\ \frac{B-C}{2} = \frac{\sin\frac{b-c}{2}}{\sin\frac{b+c}{2}}\ cotg\ \frac{A}{2}$$

The difference of longitude is A = 54° 20', b = 90°– 32° = 58° and c= 90° – 39° 38' = 50° 22' (b – c)/2 = 3° 49'; (b+c)/2 = 54° 11' and A/2 = 27° 10' see figure 2.

Rabbi Aviad Sar Shalom Basilea		Modern Calculation		
		Theoretical formula and explanation of the 1st left column	Modern calculation	Ancient formulation
tom	10.23270	log(1/cos54°11')	0.23270	10.23270
log 2	9.99930	log cos 3° 49'	– 0.000964	9.9990357
mes 2	10.28972	log cotg 27°10'	0.289717	10.289717
M	10.52172	M= $\boldsymbol{log\ tang\ \frac{B+C}{2}}$	0.521453	10.521453
tom	10.09104	log(1/sin54°11')	0.091036	10.091036
log 2	8.82324	log sin 3° 49'	– 1.176760	8.823240
mes 2	10.28972	log cotg 27°10'	0.289717	10.289717
m	9.20400	m = $log\ \boldsymbol{tang\ \frac{B-C}{2}}$	– 0.796006	9.203933

Table 5: The calculation of R. Basilea versus the modern calculation. We assumed correctly that the calculation was performed following the formulas of Napier. Furthermore we note that the ancients added 10 to the modern logarithms. Thus log 0.1 = 9 instead of – 1, log 1 = 10 instead of 0, log 10= 11 instead of 1 and log 100= 12 instead of 2. We note the exceptional precision of the manual calculation, of the trigonometric and logarithmic tables. In the first column M= $\boldsymbol{log\ tang\ \frac{B+C}{2}}$ and m = $\boldsymbol{log\ tang\ \frac{B-C}{2}}$.

The end of Basilea's calculation is written in Italian as follows:

Semisomma de angoli alle base or $\frac{B+C}{2}$ = 73° 16' instead of 73° 14' 58''
Semi differenzia de angoli alle base or $\frac{B-C}{2}$ = 9° 12' instead of 9° 05' 15''
[B] angoli maggiore (the greatest angle): B = 82° 29' instead of 82° 20' 13''

It appears that the final calculation was performed with a slight imprecision.

R. Israel Zamosc (~1700 – 1772) published his novellae *Nezah Yisrael* on the Talmud in 1741 in Frankfort on the Oder. He addressed the issue of the prayer direction in pages 52a – 52b about B. Berakhot 30b and he referred directly to *Levush ha-Tekhelet*. Although practically at the same time as R. Basilea, both rabbis were completely independent the one from the other and their reasoning was completely different.
The author noted that there are two mistakes in the explanation of Levush.

- For areas west to Israel and with the same latitude, the prayer direction is eastward with a slight deviation to the north.
- The *Levush* neglected the effect of the difference of longitude on the prayer direction. When the difference of longitude increases, at a certain moment the prayer direction which was south to the east will become eastward and then north to the east.

R. Israel Zamosc did not calculate the orientation of the prayer which is actually the important data that we look for, but he gave a very elegant and astute method to determine whether the eastward direction of prayer is deviated northward or southward. This method is illustrated by a figure in the book which is not easy to understand.
If we present the figure slightly differently, horizontally instead of vertically,[54] see figure 11, it becomes familiar and we can easily explain the method of Zamosc. He considered three cases: the towns of Tunis, Toledo and Bayonne. He considered geographical coordinates similar to those given in *Sefer Elim*. He proved that in these three towns the prayer direction is deviated slightly to the north. But he was not able to quantify this deviation; this is however of the greatest importance. In fact with modern coordinates the prayer direction in all these towns is still deviated to the south.
The data used by Zamosc are the following:

Bayonne: longitude[55] 12°, latitude 42°
Jerusalem: longitude 66°, latitude 32°

The point B is the pole of the meridian ACN passing through Bayonne, denoted by C. The length of AC is 42°, the angle B is also 42°. ACB is a spherical triangle rectangular in A and C. Thus $\cos B = \sin C \cos b = \sin 90° \cos b$.[56] Hence $B = 42°$
Now, in the spherical triangle A’C’B right-angled in A’, $\text{tang } b' = \text{tang } B * \sin c'$ where $B = 42°$ and $c' = c - 54° = 90° - 54° = 36°$. Thus $\text{tang } b' = \text{tang } 42° * \sin 36°$.
Hence $b' = 27.89° = 27° 53' < 32°$ and $b' <$ latitude of Jerusalem. Thus the point representing of Jerusalem on the meridian A’C’N of Jerusalem is between C’ and N. The great circle passing through Bayonne and Jerusalem is thus above the great circle CC’B.
Indeed the great circle CC’B is perpendicular to the meridian in C (Bayonne). The tangent in C to the great circle CC’B is also the tangent in C to the parallel of Bayonne; its direction is eastwards. Thus the great circle CC’B is tangent in C to the parallel of Bayonne. The direction of the tangent in C to the great circle CC’B is eastwards.

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[54] The drawing presented in *Nezah Yisrael* is difficult to understand. It is vertical instead of horizontal. The vertex is above. Furthermore the arc NC of the meridian NCA is not drawn and similarly the arc NC’ of the meridian NC’A’ is not drawn. The figure is incomprehensible.
[55] All the calculations are performed with the longitude 12°. However, in the beginning of this chapter the indicated longitude of Bayonne is 17°. In the beginning I thought that it was a misprint. In fact it seems that 17° was Zamosc’s longitude (to compare with 17° 30’ in *Sefer Elim*) but because of a careless mistake the rest of the calculation was performed with 12° and this mistake was not corrected. The ancients stretched already Europe in the longitude direction, but Zamosc even increased this stretching. This was the reason of the deviation of the eastwards prayer direction northwards.
[56] See appendix 3: Rectangular spherical triangles.

Conclusion. The great circle passing through Jerusalem and Bayonne is above the great circle CC'B and the prayer direction in Bayonne is northeast. If Jerusalem was exactly on C', the prayer direction would be east and if Jerusalem was between A' and C' the prayer direction would be southeast. If Jerusalem is between N and C', the great circle joining Bayonne and Jerusalem is above the great circle CC'B and the direction of the tangent in C is northeast. If Jerusalem is between C' and B, then the great circle joining Bayonne to Jerusalem is under the great circle CC'B and the direction of the tangent is southeast.

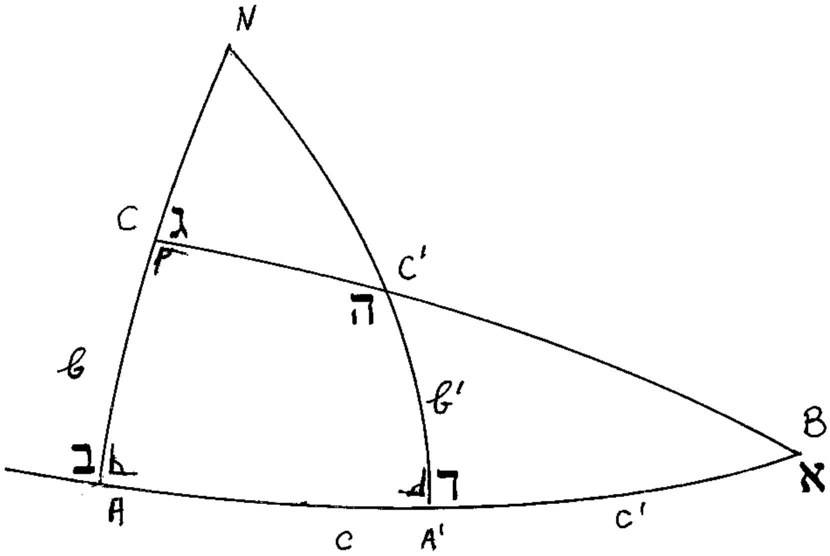

Figure 11: Method of R. Israel Zamosc. The Hebrew letters are the same as in the printed drawing and the Latin letters were chosen in order to have the angles A and A' right in order to use the classical formulas. N is the north pole, C is Bayonne, NCA is the meridian of Bayonne, NC'A' is the meridian of Jerusalem, AA'B is an arc of 90° of the equator, B is the pole of the meridian of Bayonne and CC'B is a great circle perpendicular to the meridian of Bayonne. The angles A, C, A' are right angles; b = 42°, B = 42°, c = 90°. AA' is the difference of longitude i.e. 54° and c' = 36°. J is the point representing Jerusalem; it is on the arc NC'A', the meridian of Jerusalem, either north of C', in C' or south of C'.

Jerusalem will be in C' if sin c' = tang 32° / tang 42° or if c' = 43.95° and $\Delta\lambda = 46.05°$.
Thus, at the latitude of Bayonne: if $\Delta\lambda < 46.05°$, the prayer direction is southeast.
if $\Delta\lambda = 46.05°$, the prayer direction is exactly east
if $\Delta\lambda > 46.05°$, the prayer direction is northeast.

Thus for a given latitude, when the difference of longitude between the considered location and Jerusalem increases and reaches a limit, easy to calculate, the prayer direction becomes exactly east and then it begins to be northeast. This explains why the prayer direction in America is always northeast.

In fact, when we use the modern coordinates of Bayonne we acknowledge that the difference of longitude between Bayonne and Jerusalem is only about 36.6°, therefore, b' is given by tang b' = tang 42° * sin 53.40° hence b' = 35.86° > 32°. The conclusion is reversed, Jerusalem is between A' and C' and the prayer direction is southeast.

R. Jacob Emden (1697 – 1776) raised again the problem in *Mor u- Ketsiah* Altona 1761-68, glosses on *Shulhan Arukh*. In *Orah Hayim* n° 150 he recopied the objection of Sefer *Emunat Hakhamim*[57] and mentioned also the similar objections of the astronomer R. Israel Zamosc.

R. Shneor Zalman of Liady (1745 – 1813) had a scientific culture. It came exclusively from Jewish books and especially from *Sefer Elim.*
He raised the issue in two different places, in his *Shulhan Arukh* and in his *Siddur*.
In Orah Hayim 94 he wrote:

ומקום זה בכל מדינות אלו הוא להלאה מנקודת נוכח הראש של ירושלים.[58] וצריך לחשוב כמה יהיה כנגד ירושלים ברוחב שמגלגל משוה היום[59] עד רובע עגול שמנוכח הראש שלנו[60] עד מקום פגישת[61] האופק שלנו[62] במשוה היום ואם רוחב זה שמעגול זה עד משוה היום כנגד ירושלים[63] הוא יותר מרוחב שממשוה היום עד נוכח הראש של ירושלים[64] צריך לצדד קצת לדרום כפי ערך יתרון הזה ואם הוא פחות[65] צריך לצדד לצפון קצת ודבר זה תלוי במרחק המדינה מים המערבי כלפי המזרח[66] ובהרחקה מטבור הארץ כלפי צפון[67] יותר ממרחק ירושלים. וחשבון זה קל להיודעים דרכי החשבון במשולש כדורי.

In his *Siddur*, Hilkhot Tefilin u-Keriat Shema, p. 11, he wrote:

בתפילת י"ח צריך להחזיר פניו כנגד ירושלים והמקדש. ומה שנהגו לעמוד כלפי מזרח, נתפשט המנהג מימי קדם כשהיתה הגולה בצרפת וסמוכות שלה. אבל במדינות אלו הצפוניות ביותר, צריך לעמוד כנגד קרן דרומית מזרחית ולא כנגד הקרן ממש, אלא משוך מן הקרן מעט כלפי דרום בענין שאם תחלק רובע העיגול שמאמצע הדרום לאמצע המזרח ל ג' שלישיים, יהיו פניו מכוונים כנגד רוחק ערך שליש אחד מאמצע הדרום וערך ב' שלישיים מאמצע המזרח.

Thus :

- In *Shulhan Arukh Orah Hayim* 94 he described a method allowing deciding whether the prayer direction eastwards is deviated toward the north or the south. The description of the method is unclear and unnecessarily involved. It appears that he followed exactly the method of R. Zamosc. But without a clear drawing, his explanation is incomprehensible. Especially confusing is the use of the word אופק שלנו for the great circle CC'B (see the explanatory figure of Zamosc method, fig 11). In fact this great circle is not the horizon of the location C but the tangent to this great circle is also the tangent to the parallel of C and it belongs to the horizon of the location C and its direction is W-E.[68]

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57 The agreement of R. Jacob Emden is noteworthy because in *Mitpahat Sefarim* R. Emden wrote a refutation of *Sefer Emunat Hakhamim*.
58 A'J < A'C' on the explanatory figure of the method of Zamosc. J is the point representing Jerusalem on the arc of meridian NC'A'.
59 Arc A'C'
60 Point C, our location.
61 Point B.
62 Great circle CC'B. See in the main text our commentary about the denomination אופק שלנו.
63 Arc A'C'
64 If A'C' > A'J then the prayer direction is southeast.
65 If A'C' < A'J then the prayer direction is northeast.
66 The longitude of C, hence the arc AA' corresponding to the difference of longitude between the considered location and Jerusalem.
67 The latitude AC of the considered location.
68 In the book שימת עין מאת חיים יחיאל בארנשטיין, ורשה תרע"ג, Hayim Jehiel Borenstein (1845 – 1928) was the first to identify the method of R. Shneor Zalman with that of R. Israel Zamosc. Recently R. Barukh Shovkas tried to explain the method in the local celestial sphere but his explanation remains confused and not convincing, Or Yisrael n° 28, 5762, pp.136 – 144.

- In his *Siddur*, on p. 11, he wrote that the prayer direction is toward Jerusalem. The custom to pray eastward originated, he wrote,[69] in France at the time of the Tossafot, but today in the northern countries (certainly Russia) the prayer direction is southeast according to an orientation making an angle of 60° with the east and 30° with the south corresponding to B = 150°. In the table of locations given by Delmedigo there are only two Russian towns, Lvov and Vilnius to consider if we exclude Moscow, where there lived no Jews.

  We find for these two towns the following prayer directions:

  Lvov: B = 129.36°
  Vilnius B = 149.53°

  We can thus conclude that the numerical indication given by R. Shneor Zalman refers with a very good precision to Vilnius. He was thus the only rabbi, besides R. Basilea, who made correctly the complete calculation of the prayer direction.[70] This calculation was certainly performed according to the method proposed by *Sefer Elim* using only rectangular spherical triangles. Without the help of logarithms this calculation was very difficult. The method described in his *Shulhan Arukh* was likely borrowed from the book *Nezah Israel*. Of course the result of the calculation was perverted by the imprecision and the errors of the data. This happened at the end of the 18th century, at a time when the surrounding society disposed already of precise data, but there were no possible contacts.

R. Jehiel Michal ha-Levi Epstein (1829 – 1908) published his *Arukh ha-Shulhan* during the period 1903 – 1907. He noted that only few synagogues are correctly oriented and he tried to find a justification à posteriori to this situation.[71] He referred to *Levush* and remained very careful; he spoke about "these countries" and he abstained from generalizing and extending the conclusions to America. Anyhow, he remained unclear and his statements in 94:10 and 94:14 are incomprehensible on geographical and astronomical level.[72]

R. Israel Meir ha-Kohen Kagan (1838 – 1933) published his *Mishnah Berurah* in 1884; it was universally acclaimed. He follows *Levush* in *Mishnah Berurah* on 94:2 and copied the *Levush* verbatim[73] in *Beour Halakha*. He added that in any country things should be adapted according to its situation, without any additional precision. He accepted the determination given by *Levush* for the prayer direction, based on the direction of the sun on the day of the equinox one hour after sunrise. This ruling introduces in Vilnius a still greater error than in Prague.
The recommended direction corresponds in Vilnius to B = 102.33° instead of B =158.74° (great circle) and even 162.74° if one follows the theory of the rhumb line.
The geographical error of this ruling and the general lack of precision and clarity of the author's commentary make any literal deduction of the text hazardous and risky about the prayer direction in America. However, some did not hesitate and crossed the Rubicon and

[69] With the coordinates of *Sefer Elim*, this statement is correct. However the Tossafists were not able to make such calculations. This statement is a pure assumption of Rabbi Shneor Zalman, based on his own calculations. It proves that he had a deeper knowledge of the subject than R. Israel Zamosc.
[70] Probably according to the ancient method which considered only rectangular spherical triangles.
[71] See Orah Hayim 94: 6- 9.
[72] צריך עיון גדול.
[73] However, he omitted the two words: לפום ריהטא.

decided from the text that the prayer direction in America is toward the southeast.[74] This deduction is not ingenious and passes perhaps beyond the intention of the author.

11. The kosher compass.

Since 2005, there have appeared advertisements for a kosher compass which should point toward Jerusalem and indicate the correct prayer direction.
On the website: http://www. koshercompass.com/catalog/ we find a commercial and eulogistic description of this marvelous device with the rabbinical approbation of four Israeli rabbinical authorities.[75] The principle of the working of the device is carefully hidden.
We find however on the website information about the way of calibrating the device in any new area. The calibration is performed in accordance with a table giving the calibration for a certain number of important cities in the world.
For example in New York the calibration must be made at the graduation 095[76] and in Prague it is at the graduation 139.[77]
When we compare these data with our former calculations, the working of the device becomes evident. The device is a compass in which the magnetic needle is hidden in the bottom of the device. Only an additional needle attached to the magnetic needle is visible. This visible needle must point toward Jerusalem. To that aim the device must be calibrated in each new town. The calibration consists in fixing the angle of the additional needle with respect to the magnetic needle. According to the information found on the website we learn that the angle between the two needles is locked in New York at 95° and in Prague it is locked at 139°. Surprisingly the device is thus calibrated according to the variant method of addressing the problem of facing Jerusalem. As if we could rely on the accuracy of the Mercator map.

I see several drawbacks and even problems with this device.

- This device identifies the direction of the magnetic and hidden needle to the north direction. It does not take into account the disturbing problem of the so-called magnetic variance or magnetic declination. Indeed the north magnetic pole and the north geographic pole are different. The magnetic declination is the angle between these two positions as seen from a location by the observer. The magnetic declination becomes very important near the poles and unfortunately this magnetic declination is not constant. It is variable with the time. As a result at locations close to the poles the compass readings are not very valuable unless one knows the exact magnetic declinations. The magnetic declinations can be found on the website of Natural Resources Canada.[78] For example in New York ($\lambda = -73.8°$ W, $\varphi = 40.8°$ N) the magnetic declination is 13°; 13' west with a variation of 1.7 '/year east. In Boston ($\lambda = -71°$ W, $\varphi = 42.4°$ N) the magnetic declination is 15°; 5' west with a variation of 3.8'/year east. As we can see the effects of the magnetic declination are far from being negligible. They pervert completely the indication of the device. The producers of the devise could have easily taken this phenomenon into account in their table of calibration.

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[74] See the opinion of Moishe, the inventor of the Kosher Compass who claims to be a Talmudic scholar. See http:// observantastronomer.blogspot.com/2005/11incredible-jerusalem-compass.html. See also http: //www.koshercompass.com/catalog.
[75] R. Moshe Halberstam, R. Moshe Sternbuch, R. Yosef Lieberman and R. Ya'akov Perlow.
[76] Distance 9817 km
[77] Distance 2670 km.
[78] See in French: http://www. gsc.nrcangc.ca/geomag/field/magdec-f.php
and in English: http://www. gsc.nrcangc.ca/geomag/field/magdec-eng.php

- The producer decided to calibrate the device according to the theory of the rhumb line. We consider that this choice is not judicious. The justification of this choice, based on a literal interpretation of the text of *Mishnah Berurah* and *Levush* supposed to represent the will of the Torah is certainly questionable. Furthermore, it drives off by the back of the hand the opinion the greatest authorities of the 18th century.[79] It was so easy to prepare two tables of calibration, the one according the great circle theory and the second according the rhumb line theory. It was more judicious to let the problem open and give the choice to the user.
- The calibration table is incomplete. It should include the possibility to introduce a location by its coordinates but it would require a calculation module. Anyhow it is not normal that you cannot find the calibration of the device for such important communities as Manchester and Gateshead in England.
- Because of the preceding drawbacks, the approbations given to the device by four authorities of the time are questionable. They decide in two lines of approbation to solve the problem of the prayer direction toward Jerusalem according to the rhumb line theory which they apparently ascribe beyond any doubt to R. Mordekhai Jaffe. This à posteriori attribution remains a pure assumption. They disregard superbly the opinion of such authorities as R. Basilea, R. Zamosc, R. Jacob Emden and R. Schneor Zalman. The two lasts are certainly recognized as everlasting Gedolei ha-Aharonim. These approbations should have required a detailed and nuanced conclusion. They should at least have informed the user that the device follows the theory of the rhumb lines which they ascribe to R. Mordekhai Jaffe. They should have informed the user that another opinion exists which follows the theory of the great circle. Unless they consider – but this seems not to be the case – that it is now universally granted that the *halakhah* is today according the rhumb line theory.

12. Recent halakhic developments, new trends in *Halakha*.[80]

As noted above, the two important rulers at the end of the 19th and begin of the 20th century have followed the ruling of *Levush*. Especially *Mishnah Berurah* followed closely the text of *Levush* and he copied verbatim in *Beour Halakhah* the indications given by *Levush* in order to determine practically the prayer direction.[81] However R. Israel Meir was probably not aware that this practical indication given by *Levush*, is incorrect for Prague. Furthermore the application for Vilnius and its area, of data given for Prague represents a rash generalization. Similarly he was probably not aware of the discussion whether we follow the great circle theory or the rhumb line theory and its implications. Therefore, ruling from the literal text of

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[79] R. Jacob Emden and R. Shneor Zalman. Despite their great differences, they shared the same opinion on this very specific point. Their authority still extends on our present rabbis.

[80] See the following recent publications:
Judah Herskowitz: יהודה הערשקאוויטש: בענין לאיזה צד צריך להתפלל Yeshurun Vol III, pp. 586 – 602.
Elozor Reich: Which way shall we turn? http://www.aishdas.org/articles/mizrach.htm
הרב ברוך שובקס: בירור בענין צד שכנגד א"י בתפילת שמו"ע בניו יורק, אור ישראל כ"ח תשס"ב
יהודה הערשקאוויטש: בירור בענין צד שכנגד א"י בתפילת שמו"ע בניו יורק, אור ישראל כ"ט תשס"ג
הרב יחיאל אברהם זילבר: ספר בירור הלכה תליתאה על ד' חלקי שו"ע, או"ח ס' צ"ד
Aryeh Shore: Methodologies used by Poskim to determine the orientation of the synagogue. Hakirah Vol. 11.
The first and fourth papers champion the variant solution (rhumb line), the second and sixth titles prefer the classical solution (great circle). The third and fifth titles champion the classical solution. The sixth title contains several mistakes.

[81] *Beour Halakhah* omitted two words of *Levush*. *Levush* introduced the paragraph by the two important words: **ולפום ריהטא**. These words were omitted in the transcription in *Beour Halakhah*. I think that these words are significant and prove that he did not calculate this direction.

Mishnah Berurah according to the rhumb line theory, that the prayer direction in North America, is also toward the southeast, seems certainly excessive.

Similarly the issue of a paper by Judah Herskowitz (Yeshurun Vol III, pp. 586 – 602) championing a new theory that *Levush* followed the principle of a direction toward Jerusalem along the rhumb line, seems to have exerted its influence. Indeed, it seems to be the major reference on the website Kosher Java to justify the solution of the rhumb line. On this website they propose two solutions: the tangent to the great circle ascribed to R. Aviad Sar Shalom Basilea and the tangent to the rhumb line ascribed to *Levush* on the basis of this paper.
Both solutions are presented as equally acceptable solutions. By contrast the four present rabbinical authorities who endorsed the kosher compass adopted the rhumb line theory as championed by Judah Herskowitz. It is certain that the theory of the rhumb lines fits perfectly the text of *Levush*. But this does not prove that *Levush* effectively followed this reasoning. During the 16th century only sailors and especially Portuguese sailors were acquainted with the rhumb lines. *Levush* did certainly not know them. The only way for *Levush* to know about it was the knowledge of the world map according to the projection system of Mercator. The only way for him to know the constant angle of the rhumb line with the meridians was to measure this angle on the Mercator map considering, as people did, that the straight line on the map between Prague and Jerusalem represents the shortest distance on the sphere between these two locations. Thus only the acquaintance of *Levush* with the great wall map made by Mercator in 1569 could have allowed *Levush* to know the computed direction angle. In fact we have seen that *Levush* was not able to calculate the prayer direction and that he proposed, apparently by estimation, a direction defined by B ~ 100° instead of 123° (great circle) or even 132.6° (rhumb line).[82]
This proves that *Levush* was not acquainted with the Mercator's map and could not measure the computed angle. This proves, with great likelihood, that *Levush* was not acquainted with the concept of the rhumb line. He did estimate this angle roughly and subjectively.[83]
The solution that Herskowitz ascribe to *Levush* appears to be anachronistic. *Levush* did not bring any change to the positions of R. Moses Isserles. He could only know the principle of the great circle.[84]
We must consider that the explanation that R. Jaffe added was whether oversimplified for people not accustomed to the spherical shape of the earth and its consequences, or that he was himself unaware of the influence of the effect of the difference of longitude on the prayer direction angle.
Anyhow ascribing to *Levush* the use of the theory of the rhumb lines would be an easy solution to justify his text and ensure his infallibility. But it would not solve the problem because the prayer angle of 100° would remain unjustifiable.
It is disconcerting that what appears as a pure assumption of Hershkowitz could have been accepted as a granted truth and used for adapting practical *halakhah* in contradiction with the greatest *halakhist* of the 18th century.[85] It is then surprising that it asserted itself without any

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[82] These values are calculated with the coordinates of Prague known by the ancients.
[83] He use therefore the words לפום ריהטא.
[84] Levush refers twice to those who understand the sphere.
[85] Let us even imagine that *Levush* had written that the practical prayer direction in Prague is southeastward, at equal distance from the south as from the east. In that case we would have a good argument to ascertain that *Levush* was acquainted with the Mercator map and measured the bearing angle of 135° on it. But we could not yet decide whether *Levush* was abused by the distortion of the Mercator map and did not fully understand the difference between the great circle and the rhumb line. The omission, in his explanations, of the influence of the difference of longitude with regard to Jerusalem that he made would result either from the oversimplification of his explanation or because *Levush* was fully aware of the significance of the use of the Mercator map for the measure of the bearing of the course from the chosen location to Jerusalem. Only in this last and unlikely case

opposition and reached the status of an ordinary ruling. It appears that ascribing the knowledge of the rhumb lines to *Levush* and understanding his text accordingly is a pure anachronism. It corresponds to rewriting and reinterpreting the history. Furthermore, we have seen that only the principle of the great circle makes sense.

13. Conclusions.

The prayer direction is given by the tangent in the considered location to the great circle passing through the location and Jerusalem.
This was the plain understanding of the ancients as far as they were aware of the spherical shape of the earth.
Four rabbinical authorities of the 18^{th} centuries put the emphasis on two reasoning mistakes in the explanation given by *Levush.* They accepted however his conclusions for Central Europe and none of them raised the possibility that *Levush* had followed the theory of the rhumb line.
Herskowitz proposed recently to justify the explanations of *Levush* by the use of the rhumb line theory. This would imply that *Levush* knew the Mercator map and did not understand the pervert effects of its distortion.
However the incorrect prayer direction proposed by *Levush* in Prague proves that he was not acquainted with the map of Mercator and the rhumb line.
Ascribing to *Levush* or *Mishnah Berurah* the theory of the rhumb line for the determination of the prayer direction on the globe of the earth corresponds to rewriting history and reinterpreting ancient texts giving them a new content and signification. It is thus a pure anachronism and it is unfair. The solution of the rhumb line is the result of a misunderstanding of distortion of the Mercator map and an incorrect literal exegesis of the text of *Levush*.
The quickness to adopt in practical *halakhah* the solution proposed by Hershkowitz which seems at the very most an astute assumption is surprising.
It is likely that *Levush* was not yet aware of the Mercator map and he knew only the great circle. He certainly alluded to it when mentioned twice the shape of the globe: צורת הכדור.
The adoption of a variant solution, which does not make sense, with the only aim to fit the text better, leads to anachronistic solution.
The examination of the rabbinical writings related to the issue gives very clear information about the slow development of scientific knowledge among the Jews.

Mathematical appendix.

1. Spherical Trigonometry: introduction.

Fifty years ago the spherical trigonometry was taught in the mathematic section of the secondary schools and it was a prerequisite for the entrance exam in engineering schools. Today this subject is no longer taught and most engineers graduate without any knowledge of this subject. This is mainly the justification for this short appendix and the remainder of the main formulas used.

The intersection of a sphere and a secant plane is a circle of radius r, generally smaller than the radius R of the sphere. We call it a *little circle*.

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would the reasoning of Herskowitz be founded. *Levush* would then be in opposition with his predecessors. Why then should we rule according to him?

If the secant plane contains the center O of the sphere, then the radius of the circle is r = R. This circle has the greatest possible radius; we call it a *great circle*.

If we consider two points A and B on the surface of the sphere, then the intersection of the sphere and the plane ABO is a great circle passing through A and B; this plane is unique. The points A and B define two arcs on the great circle: the one is <= 180°, the second is >= 180°.

Let us consider three points A, B and c on the surface of the sphere. There is one great circle joining A and B. Similarly, there is a great circle joining B and A and there is a great circle joining C and A.
We call *spherical triangle* the surface of the sphere delimited by three arcs of great circles joining three vertices A, B and C (see Figure 12).
In fact there are two arcs on each of these three great circles and it is possible therefore to consider 8 different surfaces delimited by three arcs chosen on these three great circles:

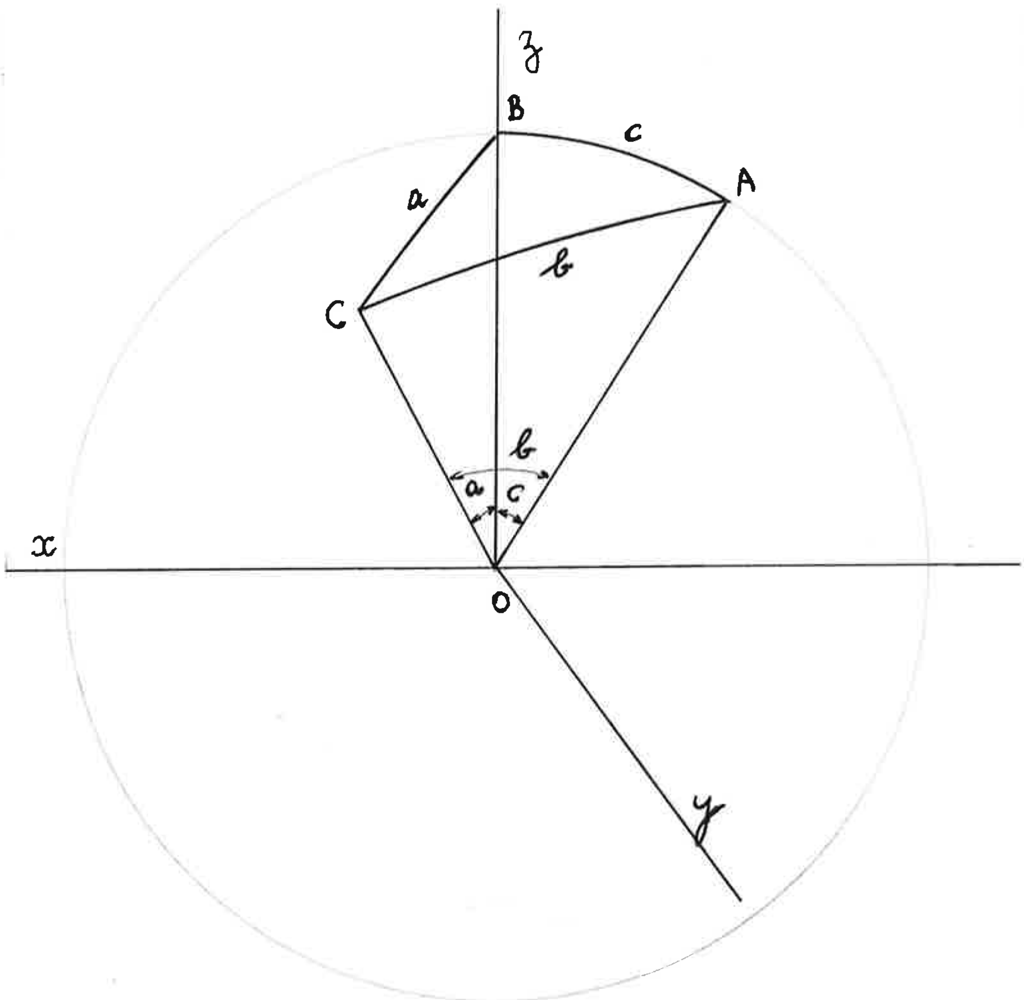

Figure 12: Representation of a sphere of center O. The spherical triangle ABC is the intersection of the sphere by the trihedron OABC. The angle A is the angle in A between the tangents in A to the great circles b and c. It is also the angle between the planes OAC and OAB. The side a, has the same measure as the central angle COB.

Three arcs < 180°: 1 triangle.
Three arcs > 180°: 1 triangle.
1 arc <180° and 2 arcs >180°: 3 triangles
2 arcs <180° and 1 arc > 180°: 3 triangles. Together there are 8 triangles.
We generally consider in spherical trigonometry only the triangle whose sides are smaller than 180°. It is the spherical triangle ABC. It is also the intersection of the surface of the sphere by the trihedron OABC with the edges OA, OB and OC.
The angle A of the spherical triangle is the angle between the tangents in A to the two great circles passing through A. This angle A is also the angle of the dihedron of edge OA defined by the two planes OAB and OAC.
The arc AB = c of the spherical triangle has the same size as the central angle AOB.
Thus the angles of the spherical triangle are also the angles of the three dihedrons of the trihedron joining the center O of the sphere to the three vertices A, B and C of the spherical triangle. The sides a, b and c of the spherical triangle have the same size as the central angles defined by the edges of the trihedron OABC.

2. Formulas of the spherical trigonometry.

For a demonstration of the following formulas see a textbook on spherical trigonometry or spherical astronomy.

System I contains 4 elements, 3 sides and 1 angle.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B.$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

System II or analogy of the sine: each relation contains 4 elements, 2 angles and 2 opposite sides.

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

System III contains 5 elements, 3 sides and 2 angles.

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A.$$
$$\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A.$$
$$\sin b \cos C = \cos c \sin a - \sin c \cos a \cos B.$$
$$\sin b \cos A = \cos a \sin c - \sin a \cos c \cos B.$$
$$\sin c \cos A = \cos a \sin b - \sin a \cos b \cos C.$$
$$\sin c \cos B = \cos b \sin a - \sin b \cos a \cos C$$

System IV contains 4 elements, 2 sides and 2 angles, one of them the inner angle. The formulas of the cotangents.

$$\cos a \cos B = \sin a \cot c - \sin B \cot C.$$
$$\cos a \cos C = \sin a \cot b - \sin C \cot B.$$
$$\cos b \cos C = \sin b \cot a - \sin C \cot A.$$
$$\cos b \cos A = \sin b \cot c - \sin A \cot C.$$
$$\cos c \cos A = \sin c \cot b - \sin A \cot B.$$

cos c cos B = sin c cot a – sin B cot A.

System I bis contains 4 elements, 3 angles and 1 side.

cos A = – cos B cos C + sin B sin C cos a.
cos B = – cos C cos A + sin C sin A cos b.
cos C = – cos A cos B + sin A sin B cos c.

System III bis contains 5 elements, 3 angles and 2 sides.

sin A cos b = cos B sin C + sin B cos C cos a.
sin A cos c = cos C sin B + sin C cos B cos a.
sin B cos c = cos C sin A + sin C cos A cos b.
sin B cos a = cos A sin C + sin A cos C cos b.
sin C cos a = cos A sin B + sin A cos B cos c.
sin C cos b = cos B sin A + sin B cos A cos c.

The formulas of Napier:

$$tang\ \frac{B+C}{2} = \frac{\cos\frac{b-c}{2}}{\cos\frac{b+c}{2}}\ cotg\ \frac{A}{2}$$

$$tang\ \frac{B-C}{2} = \frac{\sin\frac{b-c}{2}}{\sin\frac{b+c}{2}}\ cotg\ \frac{A}{2}$$

$$tang\ \frac{b+c}{2} = \frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}}\ tang\ \frac{a}{2}$$

$$tang\ \frac{b-c}{2} = \frac{\sin\frac{B-C}{2}}{\sin\frac{B+C}{2}}\ tang\ \frac{a}{2}$$

We can calculate B and C if we know b, c and A (first two formulas).
We can calculate b and c if we know B, C and a (last two formulas).

3.Rectangular spherical triangles.

If we consider a spherical triangle which is right-angled in A , then sin A = 1 and cos A = 0. The formulas of the previous subsection are translated to the following formulas:

$$\cos a = \cos b \cos c$$
$$\sin b = \sin a \sin B \qquad \sin c = \sin a \sin C$$
$$\text{tang } b = \text{tang } a \cos C \qquad \text{tang } c = \text{tang } a \cos B$$
$$\text{tang } b = \text{tang } B \sin c \qquad \text{tang } c \text{ tang } C \sin b$$
$$\cos C = \sin B \cos c \text{ and } \cos B = \sin C \cos b$$
$$\cos a = \text{cotg } B \text{ cotg } C$$

4. Historical note.

The mathematical background of the rabbis of Central and East Europe from the 17th century until the end of the 18th century was mainly *Sefer Elim* of R. Solomon Joseph Delmedigo edited in Amsterdam in 1628 by R. Manasseh ben Israel. It is thus interesting to examine which methods of calculation were available to them through this book.

Logarithms.

They were known by the publication of John Napier's book Logatithmorum Canonis Descriptio in 1614. Delmedigo mentions the logarithms, a marvelous method. " Recent scholars have found an easy method of solving any problem dealing with numbers, dispensing with complicated computations……Nowadays even a child can solve the problems of triangles….not by the aid of the planisphere and astrolabe, but by other wonderful method …….the sine or logarithmic tables".[86] Despite their mention, logarithms were not used practically in his book.

Spherical trigonometry.

The general formula of the spherical triangles was published in 1593 in the book of François Viete De Variorum. This formula is not mentioned in Sefer *Elim*.
The formulas of Napier were published in 1614 in his book. They are not mentioned in *Sefer Elim*.
Delmedigo knew only the formulas of the rectangular spherical triangle. He relied on François Viete 's book: The Canon Mathematicus published in 1579 and its tables of sinus, tangent and secant given for all the minutes of the quarter of the circle having a radius of 100,000.
This represented a great progress. In older tables, the basis radius was 60,000 and the trigonometric lines were generally given in parts or degrees, minutes and seconds. This sexagesimal notation made further calculations, for example multiplication or division of two trigonometric lines, much harder. In the same Canon Mathematicus, François Viete gave the formulas for the rectangular spherical triangles. Although he presented them as original, in fact only one of them could be considered as original. If we refer to a spherical triangle rectangular in A, the formulas:

$$\cos a = \cos b \cos c$$
$$\sin b = \sin a \sin B \qquad \sin c = \sin a \sin C$$
$$\text{tang } b = \text{tang } a \cos C \qquad \text{tang } c = \text{tang } a \cos B$$
$$\text{tang } b = \text{tang } B \sin c \qquad \text{tang } c \text{ tang } C \sin b$$

were already known by the Greeks.

[86] Yoseph Shlomo Delmedigo by Isaac Barzilay, Leiden 1974, p. 136. Sefer Elim by Demedigo, Odessa p.151.

The formulas cos C = sin B cos c and cos B = sin C cos b had been published by Geber. Finally only the formula cos a= cotg B cotg C was original.[87]

5. Orthonome and loxodrome.

We consider two locations A and B on the surface of the sphere.
The orthonome or geodesic line between A and B is the arc of the great circle joining A and B which is less or equal 180°. This arc is the shortest distance on the sphere between A and B. The bearing i.e. the angle from a reference line[88], of the orthonome changes in each point. In other words, the angle between the great circle and the northern meridian varies at each point.

The rhumb line or loxodrome is a line which crosses all the meridians of longitude at the same angle. It is also the path derived from a defined and constant bearing. On the earth, this line is the meridian through the current locations. The

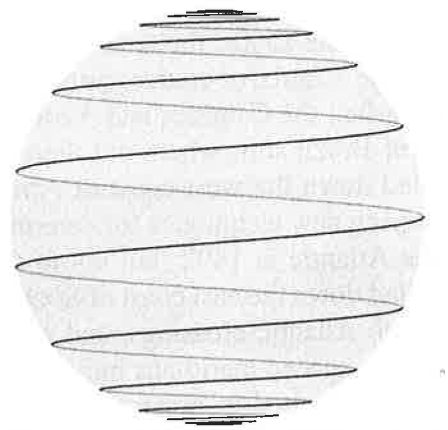

Figure 13: Representation of a rhumb line or loxodrome on a semi-transparent sphere. The bearing of this loxodrome is α = 80°; it looks like "spherical spiral". For the navigation or for the determination of the prayer direction, only a portion of the full loxodrome is relevant.

bearing is usually measured in degrees, from 0° northwards and increasing clockwise to 180° southwards, and increasing clockwise to 360° northwards again. In fact, we note that the loxodrome corresponding to a certain bearing α is the same curve as the loxodrome corresponding to the bearing α + 180°. The only difference is the orientation of the curve or in other way the direction according which the curve is covered.
The parallels cross all the meridians at straight angle. Thus all the parallels are closed loxodromes in the West –East direction (bearing 90°) or in the East – West direction (bearing 270°). All the meridians are obviously trivial loxodromes in the North – South direction (bearing 0°) and in the South – North direction (bearing 180°).
For all other bearings the rhumb line or loxodrome is an open (i.e., with two distinct ends) three dimensional curve known as *spherical helix* or *loxodromic spiral*: each end reaches the pole after an infinite number of tighter and tighter turns, see figure 15.
Thus all the loxodromes are spiral from one pole to the other. They wind round each pole an infinite number of times but reach the poles in a finite distance. The pole to pole length of a

[87] See Delambre : Histoire de l'Astronomie du Moyen-Age p. 462.
[88] The northern meridian.

loxodrome is the length of the meridian divided by the cosine of the bearing away from the north.
If we consider two points of different latitudes and longitudes, they can be connected by an infinite number of loxodromes. But one is almost always interested on the shortest, steeper one which crosses less than half the meridians. The other rhumb lines do one or more additional turns around the earth.
Under this condition there is one loxodrome joining the two points A and B on the surface of the earth. The problem to solve is finding the bearing α of this rhumb line and accessorily the

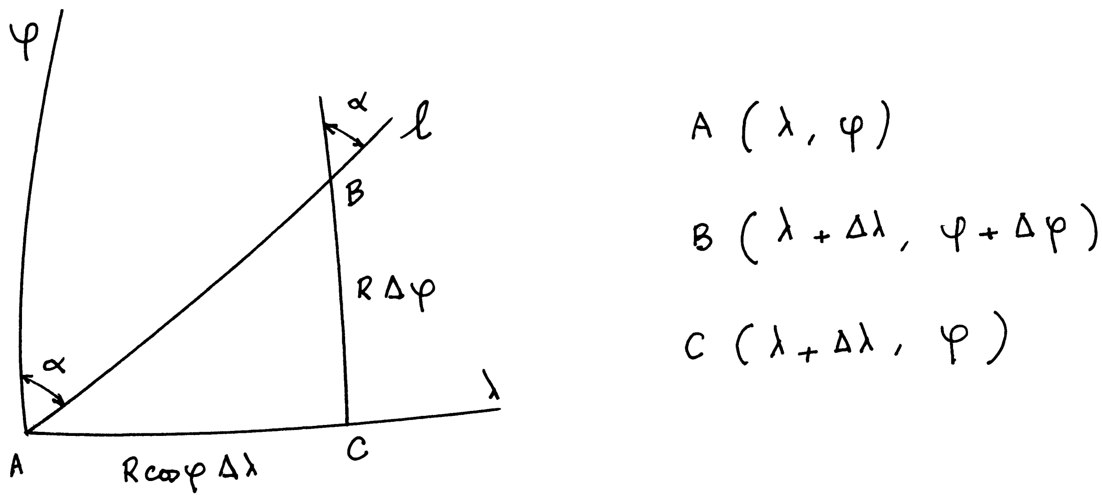

Figure 14: Equation of the loxodrome. A and B are two neighboring points of the loxodrome l.

evaluation of the length of arc AB of the loxodrme.

Calculation of the elements of the rhumb line joining two points A and B.

Let us consider two neighboring points A (λ, φ) where λ is the longitude and φ is the latitude and B (λ + Δλ, φ + Δφ) on a loxodrome l, see figure 14.
Let us consider the parallel of latitude φ with a radius r = R cos φ passing though point A. The arcs AC = r Δλ = R cos φ Δλ on the parallel and CB = R Δφ on the meridian of radius R and AB = Δs are the sides of a rectangular triangle in C. However, this triangle is not a spherical triangle as studied in spherical trigonometry, because only the side CB = R Δφ is located along a great circle.
However, if the sides of this triangle are sufficiently small, the triangle can be assimilated to a planar triangle. In this infinitely small triangle we can write:

$$tang\ \alpha = \frac{R\cos\varphi\ \Delta\lambda}{R\ \Delta\varphi}$$

$$\Delta s\cos\alpha = R\Delta\varphi$$

Hence: $\frac{\Delta\lambda}{\Delta\varphi} = \frac{tang\ \alpha}{\cos\varphi}$ and $\frac{\Delta s}{\Delta\varphi} = \frac{R}{\cos\alpha}$.

If Δφ tends to zero, then we get two differential equations:

$$\frac{d\lambda}{d\varphi} = \frac{tang\ \alpha}{\cos\varphi}$$

$$\frac{ds}{d\varphi} = \frac{R}{\cos\alpha}$$

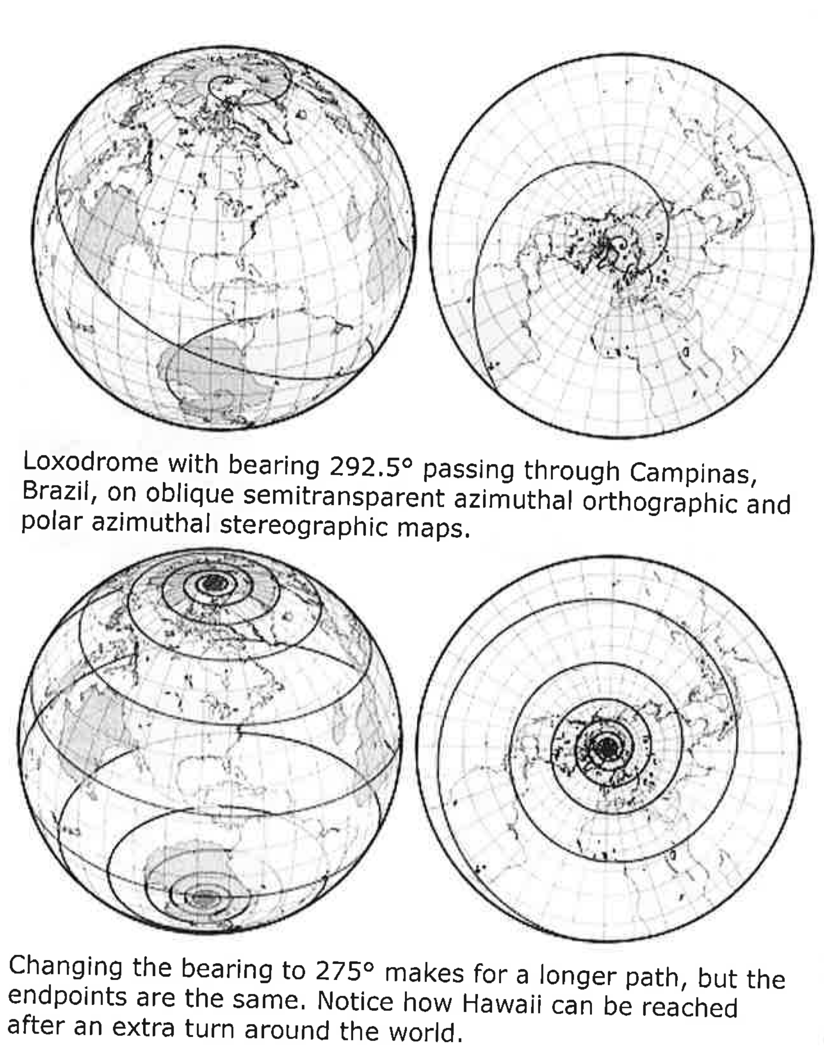

Figure 15: Two views of the same rhumb lines. The two upper figures are related to a rhumb line with a bearing of α = 112.5° or α = 292.5°. The difference between these two values is the following: the loxodrome with α = 112.5° goes from the North Pole to the South Pole, the loxodrome with α = 292.5° goes from the South Pole to the North Pole.
The two lower figures are related to a loxodrome with α = 95° or α = 275°. In the same way as we consider an arc of the great circle between a chosen location A and Jerusalem in B, we consider only an arc of the full rhumb line joining A to B. When 90° - α diminishes, the pitch of the spherical helix diminishes and the number of turns increases. When α = 90° then the loxodrome becomes the set of all the parallels.

We can easily separate the variables: $$\frac{d\varphi}{\cos\varphi} = \frac{d\lambda}{tang\ \alpha} \qquad (1)$$

$$ds = \frac{R\,d\varphi}{\cos\alpha} \qquad (2)$$

Equation (1) can be written as follows: $d\lambda = \frac{d\varphi}{\cos\varphi} * tang\ \alpha$

Hence: $\lambda = tang\ \alpha\ [\ Ln\ tang\ (\pi / 4 + \varphi / 2) + C]$

$$\lambda_2 - \lambda_1 = tang\ \alpha\ [Ln\ tang\ (\pi / 4 + \varphi_2 / 2) - Ln\ tang\ (\pi / 4 + \varphi_1 / 2)]$$
$$s = (\varphi_2 - \varphi_1)\ R / \cos\alpha$$